Some results on the relative Ritt $L^*$-order and relative Ritt $L^*$-lower order of entire functions represented by vector valued Dirichlet series

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Abstract

In this paper we introduce the idea of relative Ritt $L^*$-order and relative Ritt $L^*$-lower of entire functions represented by a vector valued Dirichlet series and study some growth properties of entire functions represented by a vector valued Dirichlet series on the basis of relative Ritt $L^*$-order and relative Ritt $L^*$-lower order.

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1 Introduction, Definitions and Notations.

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ ($\sigma$ and $t$ are real variables) defined by everywhere absolutely convergent vector valued Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \quad (1)$$

where $a_n$’s belong to a Banach space $(\mathcal{E}, \| \|)$ and $\lambda_n$’s are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}$ ($n \geq 1$), $\lambda_n \to \infty$ as $n \to \infty$ and satisfy
the conditions

$$\limsup_{n \to \infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

$$\limsup_{n \to \infty} \frac{\log\|a_n\|}{\lambda_n} = -\infty.$$ 

If $\sigma_a$ and $\sigma_c$ denote respectively the abscissa of convergence and absolute convergence of $(1)$, then in this case clearly $\sigma_a = \sigma_c = \infty$.

The function $M_f(\sigma)$ known as maximum modulus function corresponding to an entire function $f(s)$ defined by $(1)$ is written as follows:

$$M_f(\sigma) = \text{l.u.b.} \min \{f(\sigma + it) \}.$$

In the sequel the following two notations are used:

$$\log^{[k]} x = \log \left( \log^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \ldots$$

$$\log^{[0]} x = x$$

and

$$\exp^{[k]} x = \exp \left( \exp^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \ldots$$

$$\exp^{[0]} x = x.$$ 

Taking this into account, the Ritt order (See [1]) of $f(s)$, denoted by $\rho_f$, which is generally used in computational purpose, is defined in terms of the growth of $f(s)$ with respect to the exp exp z function as follows:

$$\rho_f = \limsup_{\sigma \to \infty} \frac{\log \log M_f(\sigma)}{\log \log M_{\exp\exp z}(\sigma)} = \limsup_{\sigma \to \infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$ 

Similarly, one can define the Ritt lower order of $f(s)$, denoted by $\lambda_f$ in the following manner:

$$\lambda_f = \liminf_{\sigma \to \infty} \frac{\log \log M_f(\sigma)}{\log \log M_{\exp\exp z}(\sigma)} = \liminf_{\sigma \to \infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$ 

Further an entire function $f(s)$ defined by $(1)$ is said to be of regular Ritt growth if its Ritt order coincides with its Ritt lower order. Otherwise $f(s)$ is said to be of irregular Ritt-growth.
During the past decades, several authors e.g., [1, 2, 3, 5, 7] have made intensive investigations on the properties of entire Dirichlet series related to Ritt order. Further, Srivastava [6] defined different growth parameters such as order and lower order of entire functions represented by vector valued Dirichlet series. He also obtained the results for coefficient characterization of order.

Somasundaram and Thamizharasi [8] introduced the notions of $L$-order ($L$-lower order) for entire functions where $L \equiv L(\sigma)$ is a positive continuous function increasing slowly i.e., $L(\alpha \sigma) \sim L(\sigma)$ as $\sigma \to \infty$ for every positive constant ‘$\alpha$’. In the line of Somasundaram and Thamizharasi [8], one may introduce the notion of Ritt $L$-order for an entire functions represented by vector valued Dirichlet series in the following manner:

**Definition 1** Let $f$ be an entire function represented by vector valued Dirichlet series. Then the Ritt $L$-order $\rho_f^L$ of $f$ is defined as

$$\rho_f^L = \limsup_{\sigma \to \infty} \frac{\log^2 M_f(\sigma)}{\sigma L(\sigma)}.$$  

Similarly one may define $\lambda_f^L$, the Ritt $L$-lower order of $f$ in the following way:

$$\lambda_f^L = \liminf_{\sigma \to \infty} \frac{\log^2 M_f(\sigma)}{\sigma L(\sigma)}.$$  

Further one may introduce more generalized concept of Ritt $L$-order and Ritt $L$-lower order of an entire functions represented by vector valued Dirichlet series in the following way:

**Definition 2** The Ritt $L^*$-order and Ritt $L^*$-lower order of an entire function $f$ represented by vector valued Dirichlet series are defined as

$$\rho_f^{L^*} = \limsup_{\sigma \to \infty} \frac{\log^2 M_f(\sigma)}{\sigma e^L(\sigma)} \text{ and } \lambda_f^{L^*} = \liminf_{\sigma \to \infty} \frac{\log^2 M_f(\sigma)}{\sigma e^{L(\sigma)}}$$  

respectively.

An entire function $f(s)$ defined by [1] is said to be of regular Ritt $L^*$-growth if its Ritt $L^*$-order coincides with its Ritt $L^*$-lower order. Otherwise $f(s)$ is said to be of irregular Ritt $L^*$-growth.

Srivastava [4] introduced the relative Ritt order between two entire functions represented by vector valued Dirichlet series to avoid comparing growth just with exp exp $z$ as follows:

$$\rho_g(f) = \inf \{\mu > 0 : M_f(\sigma) < M_g(\sigma \mu) \text{ for all } \sigma > \sigma_0(\mu)\}$$

$$= \limsup_{\sigma \to \infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma}.$$  

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Similarly, one can define the relative Ritt lower order of \(f(s)\) with respect to \(g(s)\), denoted by \(\lambda_g(f)\) in the following manner:

\[
\lambda_g(f) = \liminf_{\sigma \to \infty} \frac{M_g^{-1}M_f(\sigma)}{\sigma}.
\]

Extending the notion of relative Ritt order as introduced by Srivastava [4], next in this paper we introduce relative Ritt \(L^*-\)order between two entire functions represented by vector valued Dirichlet series as follows:

\[
\rho^L_g(f) = \inf \{\mu > 0 : M_f(\sigma) < M_g(\sigma e^{L(\sigma)} \mu) \text{ for all } \sigma > \sigma_0(\mu)\}
\]

\[
= \limsup_{\sigma \to \infty} \frac{M_g^{-1}M_f(\sigma)}{\sigma e^{L(\sigma)}}.
\]

Similarly, one can define the relative Ritt \(L^*-\)lower order of \(f(s)\) with respect to \(g(s)\), denoted by \(\lambda^L_g(f)\) in the following manner:

\[
\lambda^L_g(f) = \liminf_{\sigma \to \infty} \frac{M_g^{-1}M_f(\sigma)}{\sigma e^{L(\sigma)}}.
\]

For entire functions, the notions of their growth indicators such as Ritt order is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in different directions using the classical growth indicators. But at that time, the concepts of relative Ritt order of entire functions and as well as their technical advantages of not comparing with the growths of \(\exp \exp z\) are not at all known to the researchers of this area. Therefore the studies of the growths of entire functions in the light of their relative growth indicators are the prime concern of this paper. Actually in this paper we establish some newly developed results related to the growth rates of entire functions on the basis of their relative Ritt \(L^*-\)order and relative Ritt \(L^*-\)lower order.

2 Main Results.

In this section we present the main results of the paper.

**Theorem 1** Let \(f\) and \(g\) be any two entire functions represented by vector valued Dirichlet series such that \(0 \leq \lambda^L_f, \rho^L_f < \infty\) and \(0 \leq \lambda_g, \rho_g < \infty\). Then

\[
\frac{\lambda^L_f}{\rho_g} \leq \lambda^L_g(f) \leq \min \left\{ \frac{\lambda^L_f}{\lambda_g}, \frac{\rho^L_f}{\rho_g} \right\} \leq \max \left\{ \frac{\lambda^L_f}{\lambda_g}, \frac{\rho^L_f}{\rho_g} \right\} \leq \rho^L_g(f) \leq \frac{\rho^L_f}{\lambda_g}.
\]
Proof. From the definitions of $\rho^L_f$ and $\lambda^L_f$, we have for all sufficiently large values of $\sigma$ that

\[
M_f(\sigma) \leq \exp^{[2]} \left[ (\rho^L_f + \varepsilon) \sigma e^{L(\sigma)} \right],
\]

and also for a sequence of values of $\sigma$ tending to infinity, we get that

\[
M_f(\sigma) \geq \exp^{[2]} \left[ (\rho^L_f - \varepsilon) \sigma e^{L(\sigma)} \right].
\]

Similarly from the definitions of $\rho_g$ and $\lambda_g$, it follows for all sufficiently large values of $\sigma$ that

\[
M_g(\sigma) \leq \exp^{[2]} \left\{ (\rho_g + \varepsilon) \sigma \right\}
\]

\[
i.e., \quad \sigma \leq M_g^{-1}\left[ \exp^{[2]} \left\{ (\rho_g + \varepsilon) \sigma \right\} \right]
\]

\[
i.e., \quad M_g^{-1}(\sigma) \geq \left[ \frac{\log^{[2]} \sigma}{(\rho_g + \varepsilon)} \right],
\]

and for a sequence of values of $\sigma$ tending to infinity, we obtain that

\[
M_g(\sigma) \geq \exp^{[2]} \left\{ (\rho_g - \varepsilon) \sigma \right\}
\]

\[
i.e., \quad \sigma \geq M_g^{-1}\left[ \exp^{[2]} \left\{ (\rho_g - \varepsilon) \sigma \right\} \right]
\]

\[
i.e., \quad M_g^{-1}(\sigma) \leq \left[ \frac{\log^{[2]} \sigma}{(\rho_g - \varepsilon)} \right],
\]
Now from (4) and in view of (6), we get for a sequence of values of \( \sigma \) tending to infinity that

\[ M_g^{-1} M_f (\sigma) \geq M_g^{-1} \left[ \exp^{[2]} \left[ (\mu_f^* - \varepsilon) \sigma e^{L(\sigma)} \right] \right] \]

i.e., \( M_g^{-1} M_f (r) \geq \frac{\log^{[2]} \exp^{[2]} \left[ (\mu_f^* - \varepsilon) \sigma e^{L(\sigma)} \right]}{(\rho_g + \varepsilon)} \)

i.e., \( M_g^{-1} M_f (r) \geq \frac{(\mu_f^* - \varepsilon)}{(\rho_g + \varepsilon)} \sigma e^{L(\sigma)} \)

i.e., \( M_g^{-1} M_f (r) \geq \frac{(\mu_f^* - \varepsilon)}{(\rho_g + \varepsilon)} \).

As \( \varepsilon (>0) \) is arbitrary, it follows that

\[ \limsup_{\sigma \to \infty} M_g^{-1} M_f (r) \geq \frac{\rho_f^*}{\rho_g} \]

i.e., \( \rho_g^* (f) \geq \frac{\rho_f^*}{\rho_g} \). \hspace{1cm} (10)

Analogously from (3) and in view of (9), it follows for a sequence of values of \( \sigma \) tending to infinity that

\[ M_g^{-1} M_f (r) \geq M_g^{-1} \left[ \exp^{[2]} \left[ (\lambda_f^* - \varepsilon) \sigma e^{L(\sigma)} \right] \right] \]

i.e., \( M_g^{-1} M_f (r) \geq \frac{\log^{[2]} \exp^{[2]} \left[ (\lambda_f^* - \varepsilon) \sigma e^{L(\sigma)} \right]}{(\lambda_g + \varepsilon)} \)

i.e., \( M_g^{-1} M_f (r) \geq \frac{(\lambda_f^* - \varepsilon)}{(\lambda_g + \varepsilon)} \sigma e^{L(\sigma)} \)

i.e., \( M_g^{-1} M_f (r) \geq \frac{(\lambda_f^* - \varepsilon)}{(\lambda_g + \varepsilon)} \).

Since \( \varepsilon (>0) \) is arbitrary, we get from above that

\[ \limsup_{\sigma \to \infty} M_g^{-1} M_f (r) \geq \frac{\lambda_f^*}{\lambda_g} \]

i.e., \( \rho_g^* (f) \geq \frac{\lambda_f^*}{\lambda_g} \). \hspace{1cm} (11)
Again in view of (7), we have from (2) for all sufficiently large values of $\sigma$ that

$$M^-_g M_f (r) \leq M^-_g \left[ \exp^{[2]} \left[ \left( \rho^*_f + \varepsilon \right) \sigma e^{L(\sigma)} \right] \right]$$

i.e.,

$$M^-_g M_f (r) \leq \log^{[2]} \exp^{[2]} \left\{ \left( \rho^*_f + \varepsilon \right) \sigma e^{L(\sigma)} \right\} \left( \lambda_g - \varepsilon \right)$$

i.e.,

$$M^-_g M_f (r) \leq \left( \rho^*_f + \varepsilon \right) \frac{\sigma e^{L(\sigma)}}{\lambda_g - \varepsilon}.$$

Since $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\limsup_{\sigma \to \infty} M^-_g M_f (r) \leq \frac{\rho^*_f}{\lambda_g}$$

i.e.,

$$\rho^*_f (f) \leq \frac{\rho^*_f}{\lambda_g}.$$ (12)

Further from (3) and in view of (6), we get for all sufficiently large values of $\sigma$ that

$$M^-_g M_f (r) \geq M^-_g \left[ \exp^{[2]} \left[ \left( \lambda^*_f - \varepsilon \right) \sigma e^{L(\sigma)} \right] \right]$$

i.e.,

$$M^-_g M_f (r) \geq \log^{[2]} \exp^{[2]} \left\{ \left( \lambda^*_f - \varepsilon \right) \sigma e^{L(\sigma)} \right\} \left( \rho_g + \varepsilon \right)$$

i.e.,

$$M^-_g M_f (r) \geq \left( \lambda^*_f - \varepsilon \right) \frac{\sigma e^{L(\sigma)}}{\rho_g + \varepsilon}$$

i.e.,

$$M^-_g M_f (r) \geq \left( \lambda^*_f - \varepsilon \right) \frac{\sigma e^{L(\sigma)}}{\rho_g + \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\liminf_{\sigma \to \infty} \frac{M^-_g M_f (r)}{\sigma e^{L(\sigma)}} \geq \frac{\lambda^*_f}{\rho_g}$$

i.e.,

$$\lambda^*_g (f) \geq \frac{\lambda^*_f}{\rho_g}.$$ (13)
Also in view of (8), we get from (2) for a sequence of values of \( \sigma \) tending to infinity that

\[
M_g^{-1} M_f (r) \leq M_g^{-1} \left[ \exp[2] \left( (\rho_f^L + \varepsilon) \sigma e^{L(\sigma)} \right) \right]
\]

i.e., \( M_g^{-1} M_f (r) \leq \frac{\log[2] \exp[2] \left\{ (\rho_f^L + \varepsilon) \sigma e^{L(\sigma)} \right\}}{\rho_g - \varepsilon} \)

i.e., \( M_g^{-1} M_f (r) \leq \frac{\rho_f^L + \varepsilon}{\rho_g - \varepsilon} \sigma e^{L(\sigma)} \)

i.e., \( \frac{M_g^{-1} M_f (r)}{\sigma e^{L(\sigma)}} \leq \frac{\rho_f^L + \varepsilon}{\rho_g - \varepsilon} \).

Since \( \varepsilon (> 0) \) is arbitrary, we obtain from above that

\[
\liminf_{\sigma \to \infty} \frac{M_g^{-1} M_f (r)}{\sigma e^{L(\sigma)}} \leq \frac{\rho_f^L}{\rho_g} \]

i.e., \( \lambda_g^{L*} (f) \leq \frac{\rho_f^L}{\rho_g} \). (14)

Similarly from (5) and in view of (7), it follows for a sequence of values of \( \sigma \) tending to infinity that

\[
M_g^{-1} M_f (r) \leq M_g^{-1} \left[ \exp[2] \left( (\lambda_f^{L*} + \varepsilon) \sigma e^{L(\sigma)} \right) \right]
\]

i.e., \( M_g^{-1} M_f (r) \leq \frac{\log[2] \exp[2] \left\{ (\lambda_f^{L*} + \varepsilon) \sigma e^{L(\sigma)} \right\}}{\lambda_g - \varepsilon} \)

i.e., \( M_g^{-1} M_f (r) \leq \frac{\lambda_f^{L*} + \varepsilon}{\lambda_g - \varepsilon} \sigma e^{L(\sigma)} \)

i.e., \( \frac{M_g^{-1} M_f (r)}{\sigma e^{L(\sigma)}} \leq \frac{\lambda_f^{L*} + \varepsilon}{\lambda_g - \varepsilon} \).

As \( \varepsilon (> 0) \) is arbitrary, we obtain from above that

\[
\liminf_{\sigma \to \infty} \frac{M_g^{-1} M_f (r)}{\sigma e^{L(\sigma)}} \leq \frac{\lambda_f^{L*}}{\lambda_g} \]

i.e., \( \lambda_g^{L*} (f) \leq \frac{\lambda_f^{L*}}{\lambda_g} \). (15)

Thus the theorem follows from (10), (11), (12), (13), (14) and (15). □

In view of Theorem 1, one can easily deduce the following corollaries:
Corollary 1 Let \( f \) and \( g \) be any two entire functions represented by vector valued Dirichlet series such that \( 0 \leq \lambda_f^L \leq \rho_f^L < \infty \) and \( 0 \leq \lambda_g \leq \rho_g < \infty \). Then

\[
\lambda_g^L (f) = \frac{\rho_f^L}{\rho_g} \quad \text{and} \quad \rho_g^L (f) = \frac{\rho_f^L}{\lambda_g} .
\]

Corollary 2 Let \( f \) and \( g \) be any two entire functions represented by vector valued Dirichlet series such that \( 0 \leq \lambda_f^L \leq \rho_f^L < \infty \) and \( 0 \leq \lambda_g \leq \rho_g < \infty \). Then

\[
\lambda_g^L (f) = \frac{\lambda_f^L}{\rho_g} \quad \text{and} \quad \rho_g^L (f) = \frac{\lambda_f^L}{\rho_g} .
\]

Corollary 3 Let \( f \) and \( g \) be any two entire functions represented by vector valued Dirichlet series such that \( 0 \leq \lambda_f^L \leq \rho_f^L < \infty \) and \( 0 \leq \lambda_g = \rho_g < \infty \). Then

\[
\lambda_g^L (f) = \rho_g^L (f) = \frac{\rho_f^L}{\rho_g} .
\]

Corollary 4 Let \( f \) and \( g \) be any two entire functions represented by vector valued Dirichlet series such that \( 0 \leq \lambda_f^L \leq \rho_f^L < \infty \) and \( 0 \leq \lambda_g = \rho_g < \infty \). Also suppose that \( \rho_f^L = \rho_g \). Then

\[
\lambda_g^L (f) = \rho_g^L (f) = 1 .
\]

Corollary 5 Let \( f \) be an entire function represented by vector valued Dirichlet series such that \( 0 \leq \lambda_f^L \leq \rho_f^L < \infty \). Then for any entire function \( g \) represented by vector valued Dirichlet series,

(i) \( \lambda_g^L (f) = \infty \) when \( \rho_g = 0 \),
(ii) \( \rho_g^L (f) = \infty \) when \( \lambda_g = 0 \),
(iii) \( \lambda_g^L (f) = 0 \) when \( \rho_g = \infty \)

and

(iv) \( \rho_g^L (f) = \infty \) when \( \lambda_g = \infty \).

Corollary 6 Let \( g \) be an entire function represented by vector valued Dirichlet series such that \( 0 \leq \lambda_g \leq \rho_g < \infty \). Then for any entire function \( f \) represented by vector valued Dirichlet series,

(i) \( \rho_g^L (f) = 0 \) when \( \rho_f^L = 0 \),
(ii) \( \lambda_g^L (f) = 0 \) when \( \lambda_f^L = 0 \),
(iii) \( \rho_g^L (f) = \infty \) when \( \rho_f^L = \infty \)

and

(iv) \( \lambda_g^L (f) = \infty \) when \( \lambda_f^L = \infty \).
References


