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EXISTENCE AND UNIQUENESS OF SOLUTION OF THE RIEMANN-HILBERT BOUNDARY VALUE PROBLEM FOR COMPLEX PARTIAL DIFFERENTIAL EQUATION IN GENERAL CASE IN SOBOLEV SPACE

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Abstract. In this paper, we discuss on the existence and uniqueness of solution of elliptic differential equation with the Riemann-Hilbert boundary value problem

$$(1)\qquad \begin{aligned} &\frac{\partial w}{\partial \bar{z}} = F(z, w, \bar{w}, \frac{\partial w}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}}) \ in\ D \\ &Re(a+ib)w = g \ on\ \partial D \end{aligned}$$

in the Sobolev space $W_{1,p}(D)$, $2 < p < \infty$.
Then, we investigate the existence of the solution of some famous Riemann-Hilbert boundary value problems.

1. Introduction

The search of solutions to boundary value problems is of great importance, because these problems to be useful in applications, a boundary value problem should be well posed.
This means that given the input to the problem there exists a unique solution, which depends continuously on the input [1].
Much theoretical work in the field of partial differential equations is devoted to proving that boundary value problems arising from scientific and engineering applications are in fact well-posed.

Many of the most outstanding mathematicians of the twentieth century (such as Hilbert, Schwartz and Sobolev) devoted their efforts to

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study this problem and related ones.
Their works established a well-developed theory of boundary value problems for linear differential equations, and gave rise to disciplines with the modern relevance of convex analysis, monotone operators theory, distribution theory, critical point theory, Sobolev spaces, etc.
However, most phenomena in our world seem to display an intrinsically nonlinear behavior.
Thus, it became a priority to understand, as well, nonlinear problems.

In 1988, A. seif Mshimba [2, 3] studied about the Hilbert boundary value problem for elliptic differential equations

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \frac{\partial w}{\partial z}) \ in \ D$$

$$Re(a + ib)w = g \ on \ \partial D$$

in the Sobolev space $W_{1,p}(D)$.

In [4], N. Taghizadeh, M. Najand and V. Soltani discussed on solution of Hilbert boundary value problem for generalization elliptic differential equations:

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \bar{w}, \frac{\partial w}{\partial z}) + G(z, w, \bar{w}) \ in \ D$$

$$Re(a + ib)w = g \ on \ \partial D$$

in Sobolev space.

N. Taghizadeh and M. Najand Foumani in [5] discussed on existence and uniqueness of Solution of Hilbert boundary value problem for general form of elliptic differential equations:

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \bar{w}, \frac{\partial w}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}}) \tag{2}$$

in Sobolev space.

In the present work, we would like to investigate the existence and uniqueness of equation (2), with the Riemann-Hilbert boundary value problem.

2. The Riemann-Hilbert boundary value problem

Suppose that D is a domain with finite area ($S_D < \infty$) in Complex plane. And we define the weakly singular and strongly singular operators

T_D and Π_D:

$$T_D f(z) = -\frac{1}{\pi}\int\int_D \frac{f(\xi)}{\xi - z} d\zeta d\eta$$

$$\Pi_D f(z) = -\frac{1}{\pi}\int\int_D \frac{f(\xi)}{(\xi - z)^2} d\zeta d\eta$$

that $\xi = \zeta + i\eta$ and $z = x + iy$.

2.1. Existence of solution

We consider the Riemann-Hilbert boundary value problem for the given partial complex differential equation (1), i. e. we seek to find the solution w satisfying the following conditions:

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \bar{w}, \frac{\partial w}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}})\ in\ D$$

$$Re(a + ib)w = au - bv = g\ on\ \partial D$$

where $F(z, w, \bar{w}, \frac{\partial w}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}}) \in L_p(D)$. a, b and g are given real-valued measurable functions on ∂D. Moreover, $g \in W_{s,p}(D), s = 1 - \frac{1}{p}, 2 < p < \infty$, and without loss of generality, we shall assume that $a^2 + b^2 = 1$. We shall also suppose that the index of the given boundary value problem is nonnegative.

It was shown in [6], that a general solution w of the partial differential equation (2) takes the form

$$w(z) = \phi(z) + T_D F(z, w, \bar{w}, \frac{\partial w}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}}) \tag{3}$$

where ϕ is a fixed holomorphic function in D and it belongs to $W_{1,p}(D)$, $2 < p < \infty$. We exploit the arbitrariness of ϕ to solve boundary value problem.

We replace ϕ in (3) by a sum of two holomorphic functions ϕ_g and $\phi_{(w,h)}$. We thus obtain form (1)

$$Re\ (a + ib)w = Re\ (a + ib)(\phi_g + \phi_{(w,h)} + T_D F(z, w, \bar{w}, h, \bar{h}) = g\ on\ \partial D$$

then

$$Re\ (a+ib)\phi_g + Re\ (a+ib)\phi_{(w,h)} = g - Re\ (a+ib)T_D F(z, w, \bar{w}, h, \bar{h})\ on\ \partial D$$

where $h = \frac{\partial w}{\partial z}, \bar{h} = \frac{\partial \bar{w}}{\partial \bar{z}}$.

Thus our Riemann-Hilbert boundary value problem for w reduces to a similar problem for the holomorphic functions ϕ_g and $\phi_{(w,h)}$. These are

$$i. \qquad Re\ (a+ib)\phi_g = g\ on\ \partial D$$

$$ii. \qquad Re\ (a+ib)\phi_{(w,h)} = -Re\ (a+ib)T_D F(z,w,\bar{w},h,\bar{h})\ on\ \partial D.$$

Consequently;

$$i. \qquad a\ Re\ \phi_g - b\ Im\ \phi_g = g\ on\ \partial D$$

$$ii. \qquad Re\ \phi_{(w,h)} = -Re\ T_D F(z,w,\bar{w},h,\bar{h})\ on\ \partial D$$

$$iii. \qquad Im\ \phi_{(w,h)} = -Im\ T_D F(z,w,\bar{w},h,\bar{h})\ on\ \partial D.$$

Since $g \in W_{s,p}(D), s = 1 - \frac{1}{p}, p > 2$, then problem (i) has a unique solution $\phi_g \in W_{1,p}(D)$, and the following estimate hold [7, 8]:

$$\| \phi_g \|_{p,D} \leq M_1(\chi,p,D) \| g \|_{p,\partial D} + | c_0 | \sqrt[p]{S_D},$$

$$\| \phi'_g \|_{p,D} \leq M_2(\chi,p,D) \| g \|_{s,p,\partial D} .$$

With regards to problem (ii) and (iii) we note in the first instance that

$$Re\ T_D F(z,w,\bar{w},h,\bar{h}) \in W_{s,p}(D).$$

Since $F(z,w,\bar{w},h,\bar{h}) \in L_p(D), p > 2$ and as such the Riemann-Hilbert problems (ii) and (iii) have a unique solution $\phi_{(w,h)} \in W_{1,p}(D)$ and the function $\phi_{(w,h)}$ satisfies the following estimates [2, 3]:

$$\begin{aligned}
\| \phi_{(w,h)} \|_{p,D} &\leq M_3(\chi,p,D) \| F(z,w,\bar{w},h,\bar{h}) \|_{p,D} \\
\| \phi'_{(w,h)} \|_{p,D} &\leq M_2(\chi,p,D) \| Re\ T_D F(z,w,\bar{w},h,\bar{h}) \|_{s,p,\partial D} \\
&\leq M_2(\chi,p,D)\ K\ \ \| Re\ T_D F(z,w,\bar{w},h,\bar{h}) \|_{1,p,\partial D} \\
&\leq M_4(\chi,p,D) \| F(z,w,\bar{w},h,\bar{h}) \|_{p,D}
\end{aligned}$$

where χ is the index of the Riemann-Hilbert problem.

We consider the following assumptions:
I. As a function of the variables $z \in D, w, \bar{w}; F(z,w,\bar{w},\frac{\partial w}{\partial z},\frac{\partial \bar{w}}{\partial \bar{z}})$ is a continuous function of its variables.

II. The functions $F(z, w, \bar{w}, \frac{\partial w}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}})$ satisfies a Lipschitz condition of the form:

$$| F(z, w, \bar{w}, h, \bar{h}) - F(z, \widetilde{w}, \bar{\widetilde{w}}, \widetilde{h}, \bar{\widetilde{h}}) | \leq L_1 \mid w - \widetilde{w} \mid + L_2 \mid h - \widetilde{h} \mid$$

almost everywhere in D; whereas the constant L_1 is an arbitrary positive number, L_2 is strictly less than 1.

III. There exist $w, h \in L_p(D), 2 < p < \infty$, such that $F(z, w, \bar{w}, h, \bar{h}) \in L_p(D)$.
If we denote by $\Im_p(D)$ the set of pairs (w, h) for which $w, h \in L_p(D), 2 < p < \infty$, and define the norm by the relation

$$\| (w, h) \| = \| (w, h) \|_{p,\lambda} = max(\lambda \| w \|_p, \| h \|_p) \ , \ \lambda > 0$$

the set $\Im_p(D)$ is then a Banach Space.

For a pair $(w, h) \in \Im_p(D)$ we define an operator ρ as follows:

$$\rho(w, h) = (W, H)$$

$$\begin{aligned} W(z) &= \phi_g(z) + \phi_{(w,h)}(z) + T_D F(z, w, \bar{w}, h, \bar{h}) \\ H(z) &= \phi'_g(z) + \phi'_{(w,h)}(z) + \Pi_D F(z, w, \bar{w}, h, \bar{h}). \end{aligned}$$

It is an immediate consequence of the definition of the ρ that it maps $\Im_p(D)$ into itself.
Now we suppose that $(W, H), (\widetilde{W}, \widetilde{H})$ are the images of two arbitrarily chosen elements
$(w, h), (\widetilde{w}, \widetilde{h}) \in \Im_p(D)$ respectively:

$$\begin{aligned} W(z) &= \phi_g(z) + \phi_{(w,h)}(z) + T_D F(z, w, \bar{w}, h, \bar{h}) \\ H(z) &= \phi'_g(z) + \phi'_{(w,h)}(z) + \Pi_D F(z, w, \bar{w}, h, \bar{h}) \end{aligned}$$

$$\begin{aligned} \widetilde{W}(z) &= \phi_g(z) + \phi_{(\widetilde{w},\widetilde{h})}(z) + T_D F(z, \widetilde{w}, \bar{\widetilde{w}}, \widetilde{h}, \bar{\widetilde{h}}) \\ \widetilde{H}(z) &= \phi'_g(z) + \phi'_{(\widetilde{w},\widetilde{h})}(z) + \Pi_D F(z, \widetilde{w}, \bar{\widetilde{w}}, \widetilde{h}, \bar{\widetilde{h}}). \end{aligned}$$

We then obtain

$$\begin{aligned} \lambda \| W - \widetilde{W} \|_p &\leq \lambda \| \phi_{(w,h)} - \phi_{(\widetilde{w},\widetilde{h})} \|_p + \lambda \| T_D \|_p \| F(z, w, h, \bar{h}) - F(z, \widetilde{w}, \bar{\widetilde{w}}, \widetilde{h}, \bar{\widetilde{h}}) \|_p \\ &\leq \lambda \, [M_3(\chi, p, D) + B_D] \, \| F(z, w, h, \bar{h}) - F(z, \widetilde{w}, \bar{\widetilde{w}}, \widetilde{h}, \bar{\widetilde{h}}) \|_p \\ &\leq M_5(\chi, p, D) \, [L_1 + \lambda L_2] \, \| (w, h) - (\widetilde{w}, \widetilde{h}) \|_{p,\lambda} \ . \end{aligned}$$

Similarly we deduce that

$$\lambda \parallel H - \widetilde{H} \parallel_p \leq \frac{1}{\lambda} M_6(\chi, p, D) \; [L_1 + \lambda L_2] \parallel (w, h) - (\widetilde{w}, \widetilde{h}) \parallel_{p,\lambda} .$$

Consequently we have

$$\parallel (W, H) - (\widetilde{W}, \widetilde{H}) \parallel \leq max(M_5, \frac{1}{\lambda} M_6) \; [L_1 + \lambda L_2] \parallel (w, h) - (\widetilde{w}, \widetilde{h}) \parallel .$$

And if

$$[L_1 + \lambda L_2] \; max(M_5, \frac{1}{\lambda} M_6) < 1$$

then the operator ρ is contractive in $\Im_p(D)$ and, as such, by Fixed point theorem, there exists the fixed element (w, h) of the operator ρ so that

$$\rho(w, h) = (w, h)$$

that w is the solution of the given Riemann-Hilbert boundary value problem (1).

2.2. Uniqueness of solution

Suppose that

$$\begin{aligned} \rho(w, h) &= (w, h) \\ \rho(\widetilde{w}, \widetilde{h}) &= (\widetilde{w}, \widetilde{h}) \end{aligned}$$

then we have

$$\begin{aligned} \parallel (w, h) - (\widetilde{w}, \widetilde{h}) \parallel &= \parallel \rho(w, h) - \rho(\widetilde{w}, \widetilde{h}) \parallel \\ &\leq max(M_5, \frac{1}{\lambda} M_6) \; [L_1 + \lambda L_2] \parallel (w, h) - (\widetilde{w}, \widetilde{h}) \parallel \end{aligned}$$

so that

$$k = max(M_5, \frac{1}{\lambda} M_6) \; [L_1 + \lambda L_2] < 1 \tag{4}$$

and

$$\parallel (w, h) - (\widetilde{w}, \widetilde{h}) \parallel \leq k \;\; \parallel (w, h) - (\widetilde{w}, \widetilde{h}) \parallel$$

then we should have

$$\parallel (w, h) - (\widetilde{w}, \widetilde{h}) \parallel = 0$$

then

$$(w, h) = (\widetilde{w}, \widetilde{h}).$$

Consequently there exists a unique fixed element (w, h) of the operator ρ, which w is also a solution of Riemann-Hilbert boundary value problem

(1).
If we have

$$\| (w,h) - (\widetilde{w},\widetilde{h}) \| \neq 0$$

then

$$k \geq 1$$

that is opposite of the (4).

2.3. Corollary

Boundary value problem (1) contains a boundary value problem for complex partial differential equations, such as the following [9];

$$(5)\quad \begin{aligned} &\frac{\partial w}{\partial \bar{z}} - \mu_1(z)\,\frac{\partial w}{\partial z} - \mu_2(z)\,\frac{\partial \bar{w}}{\partial \bar{z}} = A(z)w + B(z)\bar{w} + C(z) \ in\ D \\ &Re(a+ib)w = g \ on\ \partial D \end{aligned}$$

when

$$F(z,w,\bar{w},\frac{\partial w}{\partial z},\frac{\partial \bar{w}}{\partial \bar{z}}) = \mu_1(z)\,\frac{\partial w}{\partial z} + \mu_2(z)\,\frac{\partial \bar{w}}{\partial \bar{z}} + A(z)w + B(z)\bar{w} + C(z)$$

in boundary value problem (1), where $A(z), B(z)$ and $C(z)$ are given complex-valued functions of complex parameter z and $\mu_1(z)$ and $\mu_2(z)$ are given $L_p(D)$ functions with $p > 2$ which fulfil the estimation

$$| \mu_1(z) + \mu_2(z) | \leq \mu_0 < 1.$$

Considering to conducted discussions boundary value problem (5) has an unique solution w in the form

$$w = \phi_g + \phi_{(w,h)} + T_D[\mu_1(z)\,\frac{\partial w}{\partial z} + \mu_2(z)\,\frac{\partial \bar{w}}{\partial \bar{z}} + A(z)w + B(z)\bar{w} + C(z)].$$

According to Riemann-Hilbert boundary value problem ϕ_g and $\phi_{(w,h)}$ should be satisfied to the following conditions:

i. $\quad a\ Re\ \phi_g - b\ Im\ \phi_g = g\ on\ \partial D$

ii. $\quad Re\ \phi_{(w,h)} = -Re\ T_D[\mu_1(z)\,\frac{\partial w}{\partial z} + \mu_2(z)\,\frac{\partial \bar{w}}{\partial \bar{z}} + A(z)w + B(z)\bar{w} + C(z)]\ on\ \partial D$

iii. $\quad Im\ \phi_{(w,h)} = -Im\ T_D[\mu_1(z)\,\frac{\partial w}{\partial z} + \mu_2(z)\,\frac{\partial \bar{w}}{\partial \bar{z}} + A(z)w + B(z)\bar{w} + C(z)]\ on\ \partial D.$

The boundary value problem (5) on the complex plane with natural restrictions on the functions $\mu_1(z), \mu_2(z), A(z),\ B(z)$ and $C(z)$ contains

many well-known boundary value problems: Cauchy-Riemann equation with Riemann-Hilbert boundary value condition, Beltrami equation with Riemann-Hilbert boundary value condition, Carleman-Bers-Vekua equation with Riemann-Hilbert boundary value condition, holomorphic disc equation with Riemann-Hilbert boundary value condition and other boundary value problems, which are obtained from (5) by an appropriate choice of the coefficients.
Now, we define some of these boundary value problems,

I. The Cauchy-Riemann equation with Riemann-Hilbert boundary value condition,

$$\frac{\partial w}{\partial \bar{z}} = 0 \ in \ D$$

$$Re(a + ib)w = g \ on \ \partial D$$

when $\mu_1(z) = \mu_2(z) = A(z) = B(z) = C(z) = 0$ in boundary value problem (5).

II. The Carlemann-Bers-Vekua equation with Riemann-Hilbert boundary value condition,

$$\frac{\partial w}{\partial \bar{z}} = A(z)w + B(z)\bar{w} + C(z) \ in \ D$$

$$Re(a + ib)w = g \ on \ \partial D$$

when $\mu_1(z) = \mu_2(z) = 0$ in boundary value problem (5).

III. The Beltrami equation with Riemann-Hilbert boundary value condition,

$$\frac{\partial w}{\partial \bar{z}} = \mu_1(z) \ \frac{\partial w}{\partial z} \ in \ D$$

$$Re(a + ib)w = g \ on \ \partial D$$

when $\mu_2(z) = A(z) = B(z) = C(z) = 0$ in boundary value problem (5).

IV. The holomorphic disc equation with Riemann-Hilbert boundary value condition,

$$\frac{\partial w}{\partial \bar{z}} = \mu_2(z)\ \frac{\partial \bar{w}}{\partial \bar{z}}\ in\ D$$

$$Re(a+ib)w = g\ on\ \partial D$$

when $\mu_1(z) = A(z) = B(z) = C(z) = 0$ in boundary value problem (5).

3. Conclusion

In this paper, we discussed the existence and uniqueness of solution of the Riemann-Hilbert boundary value problem for generalization of a first order nonlinear complex elliptic systems (1) in Sobolev space$W_{1,p}(D)$. Then, we proved that both of the Beltrami equation and the Vekua equation with Riemann-Hilbert boundary value problem have a unique solution in Sobolev space.
We offer to investigate the existence and uniqueness of solution of generalization elliptic differential equation (2) with another suitable boundary value problems.

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