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A NEW FORM OF FUZZY GENERALIZED BI-IDEALS IN ORDERED SEMIGROUPS

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Abstract. In several applied disciplines like control engineering, computer sciences, error-correcting codes and fuzzy automata theory, the use of fuzzified algebraic structures especially ordered semigroups and their fuzzy subsystems play a remarkable role. In this paper, we introduce the notion of $(\in, \in \vee \overline{q}_k)$-fuzzy subsystems of ordered semigroups namely $(\in, \in \vee \overline{q}_k)$-fuzzy generalized bi-ideals of ordered semigroups. The important milestone of the present paper is to link ordinary generalized bi-ideals and $(\in, \in \vee \overline{q}_k)$-fuzzy generalized bi-ideals. Moreover, different classes of ordered semigroups such as regular and left weakly regular ordered semigroups are characterized by the properties of this new notion. Finally, the upper part of a $(\in, \in \vee \overline{q}_k)$-fuzzy generalized bi-ideal is defined and some characterizations are discussed.

1. Introduction

Fuzzy set theory [1] is a useful tool to describe situations in which the data are imprecise or vague and handle such situations by attributing a degree to which a certain object belongs to a set. Further, utilizing this fundamental concept of fuzzy set Rosenfeld introduced the notion of fuzzy subgroup [2]. Bhakat and Das [3] generalized Rosenfeld's fuzzy group theory and defined $(\in, \in \vee q)$-fuzzy subgroup by utilizing the combine notions of "belongingness" and "quasi-coincidence" of fuzzy point

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and fuzzy set [4]. In addition, Davvaz and Khan [5] discussed some characterization regular ordered semigroups in terms of (α, β)-fuzzy generalized bi-ideals, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Moreover, in the semigroup theory the notions of generalized fuzzy (interior, bi-, left, right, quasi) ideals was studied respectively in ([6-9]). Kazanchi and Yamak [7] gave $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy bi-ideals of a semigroup and in [10] Shabir *et. al* studied $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy ideals, generalized bi-ideals and quasi-ideals of a semigroup and characterized regular semigroups by the properties of these ideals. The reader is referred to [11-21] for further study regarding (α, β)-fuzzy subsets and its generalization.

The aim of this paper to investigate more general form of fuzzy generalized bi-ideals. In this connection the notion of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideals of ordered semigroup is introduced. In addition, Characterizations of regular, left weakly regular ordered semigroups by means of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy bi-ideals are discussed. Further, the concepts of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left (right) ideals are also presented and some related properties are discussed.

2. Basic Definitions and Preliminaries

By an *ordered semigroup* (or *po-semigroup*) we mean a structure $(S, \cdot, \leq)$ in which the following conditions are satisfied:

(*OS1*) $(S, \cdot)$ is a semigroup,

(*OS2*) $(S, \leq)$ is a poset,

(*OS3*) $a \leq b \longrightarrow ax \leq bx$ and $a \leq b \longrightarrow xa \leq xb$ for all $a, b, x \in S$.

For subsets A, B of an ordered semigroup S, we denote by

$$AB = \{ab \in S \mid a \in A, b \in B\},$$
$$(A] = \{t \in S \mid t \leq h \text{ for some } h \in A\}.$$

If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For any $A, B \subseteq S$ we have $A \subseteq (A]$, $(A](B] \subseteq (AB]$ and $((A]] = (A]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$. A non-empty subset A of S is called a *left* (*right*) ideal of S if

(i) $(\forall a \in S)(\forall b \in A)$ $(a \leq b \longrightarrow a \in A)$,

(ii) $AS \subseteq A\, (SA \subseteq A)$.

A non-empty subset A of an ordered semigroup S is called a *generalized bi-ideal* [22] of S if

(i) $(\forall a \in S)(\forall b \in A)$ $(a \leq b \longrightarrow a \in A)$,

(ii) $ASA \subseteq A$.

A non-empty subset A of an ordered semigroup S is called a *bi-ideal* [23] of S if

(i) $(\forall a \in S)(\forall b \in A)$ $(a \leq b \longrightarrow a \in A)$,

(ii) $A^2 \subseteq A$,

(iii) $ASA \subseteq A$.

Note that, every bi-ideal of S is a generalized bi-ideal of S, but the converse is not true, as shown in [22].

An ordered semigroup S is *regular* [23] if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$, or equivalently, we have (i) $a \in (aSa]$ $\forall a \in S$ and (ii) $A \subseteq (ASA]$ $\forall A \subseteq S$. An ordered semigroup S is called *left* (*right*) *regular* [23] if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2 (a \leq a^2x)$, or equivalently, (i) $a \in (Sa^2](a \in (a^2S])$ $\forall a \in S$ and (ii) $A \subseteq (SA^2](A \subseteq (A^2S])$ $\forall A \subseteq S$. An ordered semigroup S is called *left* (*right*) *simple* [23] if for every left (right) ideal A of S we have $A = S$ and S is called *simple* [23] if it is both left and right simple. An ordered semigroup S is *left* (*right*) *regular* [23] if for every $a \in S$, there exists $x \in S$, such that $a \leq xa^2$ $(a \leq a^2x)$, or equivalently, (i) $a \in (Sa^2]$ $(a \in (a^2S])$ $\forall a \in S$ and (ii) $A \subseteq (SA^2]$ $(A \subseteq (A^2S])$ $\forall A \subseteq S$. An ordered semigroup S is called *completely regular* [23] if it is left regular, right regular and regular. An ordered semigroup S is called *left weakly regular* [22] if for every $a \in S$, there exist $x, y \in S$ such that $a \leq xaya$, or equivalently, (i) $a \in ((Sa)^2]$ $\forall a \in S$ and (ii) $A \subseteq ((SA)^2]$ $\forall A \subseteq S$. *Right weakly regular* ordered semigroups are defined similarly. An ordered semigroup S is called *weakly regular* if it is both a left and right weakly regular.

Note that if S is commutative, then the concepts of regular and weakly regular ordered semigroups coincide.

By $B(a)$ $(L(a), R(a)$ and $I(a))$ we mean the generalized bi-(left, right and two-sided) ideal of S generated by a $(a \in S)$ denoted by

$$\begin{aligned} B(a) &= (a \cup aSa], L(a) = (a \cup Sa], R(a) = (a \cup aS], \\ I(a) &= (a \cup Sa \cup aS \cup SaS] \text{ (see [22, 23])}. \end{aligned}$$

Now, we give some fuzzy logic concepts.

A function $F : S \longrightarrow [0,1]$ (unit closed interval) is called a *fuzzy subset* of S.

The study of fuzzification of algebraic structures has started in the pioneering paper of Rosenfeld [3] in 1971. Rosenfeld introduced the notion of fuzzy groups and successfully extended many results from groups in the theory of fuzzy groups. Kuroki [24] studied fuzzy ideals, fuzzy bi-ideals and semiprime fuzzy ideals in semigroups (also see [25-27]).

If F and G are fuzzy subsets of S then $F \preceq G$ means $F(x) \leq G(x)$ for all $x \in S$ and the symbols $\wedge$ and $\vee$ will mean the following fuzzy subsets are defined as follow for all $x \in S$:

$$F \wedge G \quad : \quad S \longrightarrow [0,1] | x \longmapsto (F \wedge G)(x) = F(x) \wedge G(x) = \min\{F(x), G(x)\},$$
$$F \vee G \quad : \quad S \longrightarrow [0,1] | x \longmapsto (F \vee G)(x) = F(x) \vee G(x) = \max\{F(x), G(x)\}.$$

A fuzzy subset F of S is called a *fuzzy subsemigroup* if

$$F(xy) \geq \min\{F(x), G(y)\} \text{ for all } x, y \in S.$$

A fuzzy subset F of S is called a *fuzzy generalized bi-ideal* [22] of S if:
(i) $x \leq y \longrightarrow F(x) \geq F(y)$,
(ii) $F(xyz) \geq \min\{F(x), F(z)\}$ for all $x, y, z \in S$.

A fuzzy subset F of S is called a *fuzzy left (right)-ideal* [23] of S if:
(i) $x \leq y \longrightarrow F(x) \geq F(y)$,
(ii) $F(xy) \geq F(y)\,(F(xy) \geq F(x))$ for all $x, y \in S$.

A fuzzy subset of S is called a *fuzzy ideal* if it is both a fuzzy left and right ideal of S.

A fuzzy subsemigroup F is called a *fuzzy bi-ideal* [23] of S if:
(i) $x \leq y \longrightarrow F(x) \geq F(y)$,
(ii) $F(xyz) \geq \min\{F(x), F(z)\}$ for all $x, y, z \in S$.

Note that every fuzzy bi-ideal is a generalized fuzzy bi-ideal of S. But the converse is not true, as given in [22].

Let F be a fuzzy subset of an ordered semigroup S, then for all $t \in (0,1]$, the set $U(F;t) = \{x \in S | F(x) \geq t\}$ is called a *level set* of F.

Theorem 2.1. ([28]) *A fuzzy subset F of an ordered semigroup S is a fuzzy left (right)-ideal of S if and only if $U(F;t)\,(\neq \emptyset)$ where $t \in (0,1]$ is a left (right)-ideal of S.*

Theorem 2.2. ([5]) *A fuzzy subset F of an ordered semigroup S is a fuzzy generalized bi-ideal of S if and only if $U(F;t)\,(\neq \emptyset)$ where $t \in (0,1]$ is a generalized bi-ideal of S.*

Theorem 2.3. ([28]) *A non-empty subset A of an ordered semigroup S is a left (right)-ideal of S if and only if*

$$\chi_A : S \longrightarrow [0,1] | x \longmapsto \chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

is a fuzzy left (right)-ideal of S.

Theorem 2.4. ([5]) *A non-empty subset A of an ordered semigroup S is a generalized bi-ideal of S if and only if*

$$\chi_A : S \longrightarrow [0,1] | x \longmapsto \chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

is a fuzzy generalized bi-ideal of S.

If $a \in S$ and A is a non-empty subset of S. Then,

$$A_a = \{(y,z) \in S \times S \,|\, a \leq yz\}.$$

For any two fuzzy subsets F and G of an ordered semigroup S, the product $F \circ G$ is defined by:

$$F \circ G : S \longrightarrow [0,1] | a \longmapsto (F \circ G)(a) = \begin{cases} \bigvee_{(y,z)\in A_a} (F(y) \wedge G(z)), & \text{if } A_a \neq \emptyset, \\ 0, & \text{if } A_a = \emptyset. \end{cases}$$

Let F be a fuzzy subset of S, then the set of the form:

$$F(y) := \begin{cases} t \in (0,1], & \text{if } y = x, \\ 0, & \text{otherwise}, \end{cases}$$

is called a *fuzzy point* with support x and value t and is denoted by $(x;t)$ [2]. A fuzzy point $(x;t)$ is said to *belong to* (*quasi-coincident* with) a fuzzy set F, written as $(x;t) \in F$ ($(x;t)\mathrm{q}F$) if $F(x) \geq t$ ($F(x) + t > 1$) [2]. If $(x;t) \in F$ or $(x;t)\mathrm{q}F$, then we write $(x;t) \in \vee\mathrm{q}F$. The symbol $\overline{\in \vee \mathrm{q}}$ means $\in \vee\mathrm{q}$ does not hold.

Generalizing the concept of $(x;t)\mathrm{q}F$, in semigroups, Khan *et. al.* [9] defined $(x;t)\mathrm{q}_kF$, as $F(x) + t + k > 1$, where $k \in [0,1)$.

Throughout in this paper S will denote an ordered semigroup unless otherwise specified.

3. $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideals

In this section, we define a more generalized form of (α,β)-fuzzy generalized bi-ideals of an ordered semigroup S and introduce $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideals of S, where $\alpha \in \{\overline{\in}, \overline{\mathrm{q}}, \overline{\in} \wedge \overline{\mathrm{q}}, \overline{\in} \vee \overline{\mathrm{q}}\}$ and k is an arbitrary element of $[0,1)$ unless otherwise specified.

Definition 3.1. A fuzzy subset F of S is called a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-*fuzzy subsemigroup* of S if for all $x, y \in S$ and $t \in (0,1]$ the following holds:

$$(xy;t)\,\overline{\in} F \longrightarrow (x;t)\,\overline{\in} \vee \overline{\mathrm{q}}_k F \text{ or } (y;t)\,\overline{\in} \vee \overline{\mathrm{q}}_k F.$$

Definition 3.2. A fuzzy subset F of S is called a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-*fuzzy generalized bi-ideal* of S if for all $x, y, z \in S$ and $t \in (0,1]$ the following conditions hold:
(1) $(\forall x \leq y)$, $(x;t)\,\overline{\in} F \longrightarrow (y;t)\,\overline{\in} \vee \overline{q}_k F$,
(2) $(xyz;t)\,\overline{\in} F \longrightarrow (x;t)\,\overline{\in} \vee \overline{q}_k F$ or $(y;t)\,\overline{\in} \vee \overline{q}_k F$.

Lemma 3.3. *For any fuzzy subset F of an ordered semigroup S and for all $x, y \in S$ and $t \in (0,1]$ the following conditions are equivalent:*
$(1a)$ $(\forall x \leq y)$, $(x;t)\,\overline{\in} F \longrightarrow (y;t)\,\overline{\in} \vee \overline{q}_k F$,
(1b) $(\forall x \leq y)$, $\max\left\{F(x), \frac{1-k}{2}\right\} \geq F(y)$.

Proof. $(1a) \Longrightarrow (1b)$: Suppose that there exist $x, y \in S$ with $x \leq y$ such that

$$\max\left\{F(x), \frac{1-k}{2}\right\} < F(y),$$

then

$$\max\left\{F(x), \frac{1-k}{2}\right\} < t \leq F(y) \text{ for some } t \in \left(\frac{1-k}{2}, 1\right].$$

Shows that $(x;t)\,\overline{\in} F$ but $(y;t) \in \wedge q_k F$, a contradiction. Therefore we accept that

$$\max\left\{F(x), \frac{1-k}{2}\right\} \geq F(y) \text{ for all } x, y \in S, \text{with } x \leq y.$$

$(1b) \Longrightarrow (1a)$: Let $(x;t)\,\overline{\in} F$ for all $x, y \in S$ such that $x \leq y$ and $t \in (0,1]$, then $F(x) < t$. If

$$\max\left\{F(x), \frac{1-k}{2}\right\} = F(x),$$

then $F(y) \leq F(x) < t$ follows that $(y;t)\,\overline{\in} F$. On the other hand if

$$\max\left\{F(x), \frac{1-k}{2}\right\} = \frac{1-k}{2},$$

then $F(y) \leq \frac{1-k}{2}$. Suppose that $(y;t) \in F$, then $t \leq F(y) \leq \frac{1-k}{2}$ follows that $(y;t)\,\overline{q} F$ consequently $(y;t)\,\overline{\in} \vee \overline{q} F$. □

Lemma 3.4. *Let F be a fuzzy subset of an ordered semigroup S. Then the following conditions are equivalent for all $x, y \in S$ and $t \in (0,1]$:*
(2a) $(xy;t)\,\overline{\in} F \Longrightarrow (x;t)\,\overline{\in} \vee \overline{q}_k$ *or* $(y;t)\,\overline{\in} \vee \overline{q}_k$,
(2b) $\max\left\{F(xy), \frac{1-k}{2}\right\} \geq \min\left\{F(x), F(y)\right\}$.

Proof. $(2a) \Longrightarrow (2b)$: Let $\max\left\{F(xy), \frac{1-k}{2}\right\} < \min\{F(x), F(y)\}$ for some $x, y \in S$, then

$$\max\left\{F(xy), \frac{1-k}{2}\right\} < t \leq \min\{F(x), F(y)\} \text{ for some } t \in \left(\frac{1-k}{2}, 1\right].$$

shows that $(xy; t)\overline{\in}F$ but $(x; t) \in \wedge q_k F$ and $(y; t) \in \wedge q_k F$ a contradiction. Hence

$$\max\left\{F(xy), \frac{1-k}{2}\right\} \geq \min\{F(x), F(y)\} \text{ for all } x, y \in S.$$

$(2b) \Longrightarrow (2a)$: If $(xy; t)\overline{\in}F$ for all $x, y \in S$ such that $x \leq y$ and $t \in (0, 1]$, then $F(xy) < t$. Consider

$$\max\left\{F(xy), \frac{1-k}{2}\right\} = F(xy),$$

then

$$\min\{F(x), F(y)\} \leq \max\left\{F(xy), \frac{1-k}{2}\right\} = F(xy) < t,$$

follows that $(x; t)\overline{\in}F$ or $(y; t)\overline{\in}F$. But if

$$\max\left\{F(xy), \frac{1-k}{2}\right\} = \frac{1-k}{2},$$

then

$$\min\{F(x), F(y)\} \leq \max\left\{F(xy), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

Suppose that $(x; t) \in F$ or $(y; t) \in F$. Then $t \leq F(x) < \frac{1-k}{2}$ or $t \leq F(y) < \frac{1-k}{2}$ follows that $F(x) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ or $F(y) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$. Hence $(x; t)\,\overline{q}_k F$ or $(y; t)\,\overline{q}_k F$. Thus $(x; t)\,\overline{\in} \vee \overline{q}_k F$ or $(y; t)\,\overline{\in} \vee \overline{q}_k F$. □

Lemma 3.5. *Let F be a fuzzy subset of an ordered semigroup S. Then for all $x, y, z \in S$ and $t \in (0, 1]$, the following conditions are equivalent:*

(3a) $(xyz; t)\,\overline{\in}F \Longrightarrow (x; t)\,\overline{\in} \vee \overline{q}_k F$ *or* $(z; t)\,\overline{\in} \vee \overline{q}_k F$,

(3b) $\max\left\{F(xyz), \frac{1-k}{2}\right\} \geq \min\{F(x), F(z)\}$.

Proof. Follows from the proofs of Lemma 3.3 and 3.4. □

From Lemma 3.3 and 3.5, we have the following theorem:

Theorem 3.6. *A fuzzy subset F of S is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S if and only if it satisfies the condition (1b) and (3b).*

Definition 3.7. A fuzzy subset F which is both a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal and a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy subsemigroup of S is called a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-*fuzzy bi-ideal* of S.

From Lemma 3.3, 3.4 and 3.5, we have the following theorem:

Theorem 3.8. *A fuzzy subset F of S is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy bi-ideal of S if and only if it satisfies the condition (1b), (2b) and (3b).*

Theorem 3.9. *Let F be a fuzzy subset of an ordered semigroup S. Then F is $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S if and only if*

$$U(F;t) = \{x \in S \mid F(x) \geq t\} \neq \emptyset$$

is generalized bi-ideal of S for all $t \in \left(\frac{1-k}{2}, 1\right]$.

Proof. Let F be $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S and $t \in \left(\frac{1-k}{2}, 1\right]$ be such that $U(F;t) \neq \emptyset$. Then by Lemma 3.3 (1b),

$$F(b) \leq \max\left\{F(a), \frac{1-k}{2}\right\} \text{ for } a \leq b \in U(F;t),$$

follows that $t \leq F(b) \leq \max\left\{F(a), \frac{1-k}{2}\right\}$ that is $F(a) \geq t$ (as $t \in \left(\frac{1-k}{2}, 1\right]$) hence $a \in U(F;t)$.

Next, we let $a, c \in U(F;t)$, then by Lemma 3.5 (3b),

$$\max\left\{F(abc), \frac{1-k}{2}\right\} \geq \min\{F(a), F(c)\} \geq \min\{t, t\} = t.$$

follows that $F(abc) \geq t$ (as $t \in \left(\frac{1-k}{2}, 1\right]$), hence $abc \in U(F;t)$.

Conversely, let $U(F;t) = \{x \in S \mid F(x) \geq t\} \neq \emptyset$ be a generalized bi-ideal of S for all $t \in \left(\frac{1-k}{2}, 1\right]$. Let $a, b \in S$ with $a \leq b$ such that $F(b) > \max\left\{F(a), \frac{1-k}{2}\right\}$, then $F(b) \geq t_0 > \max\left\{F(a), \frac{1-k}{2}\right\}$ for some $t_0 \in \left(\frac{1-k}{2}, 1\right]$. This shows that $b \in U(F;t_0)$ but $a \overline{\in} U(F;t_0)$, a contradiction and hence $F(x) \leq \max\left\{F(y), \frac{1-k}{2}\right\}$ for all $x \leq y$. □

Example 3.10. Consider the ordered semigroup $S = \{a, b, c, d\}$ with the following multiplication table "·" and order relation "≤":

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}.$$

Then $\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,c,d\}$ and $\{a,b,c,d\}$ are generalized bi-ideals of S. However, $\{a,c\}, \{a,d\}$ and $\{a,c,d\}$ are not bi-ideals of S. Define a fuzzy subset F of S as follows:

$$F : S \longrightarrow [0,1] | x \longmapsto F(x) = \begin{cases} 0.50, & \text{if } x = a, \\ 0.10, & \text{if } x = b, \\ 0.30, & \text{if } x = c, \\ 0.40, & \text{if } x = d. \end{cases}$$

Then

$$U(F;t) = \begin{cases} S, & \text{if } 0.00 < t \leq 0.10, \\ \{a,c,d\}, & \text{if } 0.10 < t \leq 0.30, \\ \{a,d\}, & \text{if } 0.30 < t \leq 0.40, \\ \{a\}, & \text{if } 0.40 < t \leq 0.50, \\ \emptyset, & \text{if } 0.50 < t \leq 1.00. \end{cases}$$

Thus, by Theorem 3.9, F is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal of S for all $t \in \left(\frac{1-k}{2}, 1\right]$ with $k = 0.3$.

Note that, every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy bi-ideal of S is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal of S. However, the converse is not true, in general, as shown in the following example.

Example 3.11. Consider the ordered semigroup as shown in Example 3.10. Then F is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal of S but F is not a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy bi-ideal of S because by condition (2b) of Lemma 3.4, we have

$$\max\left\{F(xy), \frac{1-k}{2}\right\} \geq \min\{F(x), F(y)\}.$$

If $x = y = d$ and $k = 0.3$ then

$$\max\left\{F(dd) = 0.1, \frac{1-k}{2} = 0.35\right\} \ngeq F(d) = 0.4.$$

Proposition 3.12. *If F is $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S and*

$$F_{\frac{1-k}{2}} = \left\{a \in S | F(a) > \frac{1-k}{2}\right\},$$

then $F_{\frac{1-k}{2}}$ is a generalized bi-ideal of S.

Proof. Let $a, b \in S$ such that $a \leq b \in F_{\frac{1-k}{2}}$. Then by Lemma 3.3 (1b),

$$\max\left\{F(a), \frac{1-k}{2}\right\} \geq F(b) > \frac{1-k}{2}.$$

This implies $F(a) > \frac{1-k}{2}$ (since $\frac{1-k}{2} \ngtr \frac{1-k}{2}$) i.e. $a \in F_{\frac{1-k}{2}}$.

Next, we let $a, b, c \in S$ such that $a, c \in F_{\frac{1-k}{2}}$. Then by Lemma 3.5 (3b),

$$\begin{aligned}\max\left\{F(abc), \frac{1-k}{2}\right\} &\geq \min\{F(a), F(c)\} \\ &> \frac{1-k}{2}.\end{aligned}$$

From this we see that $F(abc) > \frac{1-k}{2}$ (since $\frac{1-k}{2} \ngtr \frac{1-k}{2}$) and we write $abc \in F_{\frac{1-k}{2}}$. Hence $F_{\frac{1-k}{2}}$ is a generalized bi-ideal of S. □

Consider a fuzzy subset F of S and $t \in (0, 1]$. We define two sets as in the following:

$Q^k(F; t) = \{x \in S \mid [x; t]\, q_k F\}$ and $[F]_t^k = \{x \in S \mid [x; t] \in \vee q_k F\}$.

Theorem 3.13. *If F is $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S and*

$$Q^k(F; t) = \{x \in S \mid [x; t]\, q_k F\} \neq \emptyset,$$

then $Q^k(F; t)$ is generalized bi-ideal of S for all $t \in \left(\frac{1-k}{2}, 1\right]$.

Proof. Let F be $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S. Let $a, b \in S$ such that $a \leq b \in Q^k(F; t)$ and $t \in \left(\frac{1-k}{2}, 1\right]$, then by Lemma 3.3 (1b)

$$\max\left\{F(a), \frac{1-k}{2}\right\} \geq F(b) > 1-k-t > (1-k) - \left(\frac{1-k}{2}\right) = \frac{1-k}{2}.$$

This shows $F(a) > 1-k-t$ and we write $a \in Q^k(F; t)$.

Next, we let $a, b, c \in S$ such that $a, c \in Q^k(F; t)$, then by Lemma 3.5 (3b),

$$\begin{aligned}\max\left\{F(abc), \frac{1-k}{2}\right\} &\geq \min\{F(a), F(c)\} \\ &> \min\{1-k-t, 1-k-t\} \\ &= 1-k-t.\end{aligned}$$

This shows that $F(abc) > 1-k-t$ (as $1-k-t > \frac{1-k}{2}$) and we write $abc \in Q^k(F; t)$. Hence $Q^k(F; t)$ is generalized bi-ideal of S. □

From Theorem 3.9 and Theorem 3.13 we can prove the following Theorem.

Theorem 3.14. *Let F be a fuzzy subset of S. Then F is $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S if and only if $[F]_t^k \neq \emptyset \Rightarrow [F]_t^k$ is a generalized bi-ideal of S for all $t \in \left(\frac{1-k}{2}, 1\right]$.*

Proposition 3.15. *Every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of a regular ordered semigroup S is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy bi-ideal of S.*

Proof. Let $a, b \in S$ and F be a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S. Since S is regular, there exists $x \in S$ such that $b \leq bxb$. Then

$$\max\left\{F(ab), \frac{1-k}{2}\right\} = F(ab) \geq F(a(bxb)) = F(a(bx)b) \geq \{F(a) \wedge F(b)\}.$$

This means that F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy subsemigroup of S. Thus F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy bi-ideal of S. □

Proposition 3.16. *Every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of a left weakly regular ordered semigroup S is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy bi-ideal of S.*

Proof. Let $a, b \in S$ and F be a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S. Since S is left weakly regular, there exist $x, y \in S$ such that $b \leq xbyb$. Then

$$\max\left\{F(ab), \frac{1-k}{2}\right\} = F(ab) \geq F(a(xbyb)) = F(a(xby)b) \geq \{F(a) \wedge F(b)\}.$$

This means that F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy subsemigroup of S. Thus F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy bi-ideal of S. □

Remark 3.17. From Proposition 3.15 and 3.16, it follows that in regular and left weakly regular ordered semigroups, the concepts of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy bi-ideals coincide.

Lemma 3.18. *A non empty subset A of S is generalized bi-ideal of S if and only if the characteristic χ_A of A is $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S.*

Proof. The proof is obvious and is omitted. □

Definition 3.19. A fuzzy subset F of S is called $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left (right) ideal of S if

(i) $(\forall x, y \in S, t \in (0,1]$ and $x \leq y)$ $(x;t)\,\overline{\in} F \rightarrow (y;t)\,\overline{\in} \vee \overline{q}_k F$,

(ii) $(\forall x, y \in S, t \in (0,1])$ $(xy;t)\,\overline{\in} F \rightarrow (y;t)\,\overline{\in} \vee \overline{q}_k F$ $((x;t)\,\overline{\in} \vee \overline{q}_k F)$.

Theorem 3.20. *A fuzzy subset F of S is $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left (right) ideal of S if and only if the following conditions hold for all $x, y \in S$, $t \in (0,1]$:*
(iii) $\max\left\{F(x), \frac{1-k}{2}\right\} \geq F(y)$ *for all* $x \leq y$.
(iv) $\max\left\{F(xy), \frac{1-k}{2}\right\} \geq F(y)$ $\left(\max\left\{F(xy), \frac{1-k}{2}\right\} \geq F(x)\right)$.

Proof. Let F be $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left ideal of S and

$$\max\left\{F(a), \frac{1-k}{2}\right\} < F(b) \text{ for some } a, b \in S \text{ with } a \leq b.$$

Then

$$\max\left\{F(a), \frac{1-k}{2}\right\} < t \leq F(b) \text{ for some } t \in \left(\frac{1-k}{2}, 1\right].$$

This show that $(a;t)\overline{\in}F$ but $(b;t) \in F$, a contradiction and hence

$$\max\left\{F(x), \frac{1-k}{2}\right\} \geq F(y) \text{ for all } x \leq y.$$

Next, we consider

$$\max\left\{F(ab), \frac{1-k}{2}\right\} < F(b) \text{ for some } a, b \in S.$$

Then there exist some $t \in \left(\frac{1-k}{2}, 1\right]$ such that

$$\max\left\{F(ab), \frac{1-k}{2}\right\} < t \leq F(b).$$

We see that $(ab;t)\overline{\in}F$ but $(b;t) \in F$, a contradiction and thus

$$\max\left\{F(xy), \frac{1-k}{2}\right\} \geq F(y).$$

Conversely, let (iii) and (iv) are satisfied for all $x, y \in S$. Let $x, y \in S$ with $x \leq y$ such that $(x;t)\overline{\in}F$. Then by (iii)

$$\begin{aligned} F(y) &\leq \max\left\{F(x), \frac{1-k}{2}\right\} \\ &= \begin{cases} F(x) < t, & \text{if } F(x) \geq \frac{1-k}{2}, \\ \frac{1-k}{2}, & \text{if } F(x) < \frac{1-k}{2}. \end{cases} \end{aligned}$$

From here we observe that $(y;t)\overline{\in}F$ or $F(y) + t + k < 1$ (if $t \leq \frac{1-k}{2}$) i.e. $(y;t)\,\overline{q}_kF$. On the other hand, if $t > \frac{1-k}{2}$ then $(y;t)\overline{\in}F$ and consequently $(y;t)\,\overline{\in} \vee \overline{q}_kF$.

Lastly we choose $x, y \in S$ such that $(xy; t)\,\overline{\in} F$, then by (iv),

$$F(y) \leq \max\left\{F(xy), \frac{1-k}{2}\right\} = \begin{cases} F(xy) < t, & \text{if } F(xy) \geq \frac{1-k}{2}, \\ \frac{1-k}{2}, & \text{if } F(xy) < \frac{1-k}{2}. \end{cases}$$

It follows that $(y; t)\,\overline{\in} F$ or $F(y) + t + k < 1$ (if $t \leq \frac{1-k}{2}$) i.e. $(y; t)\,\overline{q}_k F$. On the other hand, if $t > \frac{1-k}{2}$ then $(y; t)\,\overline{\in} F$. Hence $(y; t)\,\overline{\in} \vee \overline{q}_k F$. Consequently, F is $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left ideal of S. Similarly one can show this for right ideal. □

Theorem 3.21. *A fuzzy subset λ of S is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left (right)-ideal of S if and only if $U(\lambda; t)\,(\neq \emptyset)$ is a left (right)-ideal of S for all $t \in \left(0, \frac{1-k}{2}\right]$.*

Proof. The proof follows from Theorem 3.9. □

4. Upper parts of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideals

In this section, we define the upper/lower parts of an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal and characterize regular and left weakly regular ordered semigroups in terms of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left (right)-ideals.

Definition 4.1. ([10]) For any fuzzy subsets F and G of S, then for all $x \in S$ the fuzzy subsets F^{+k}, $\left(F \wedge^k G\right)^+$, $(F \vee^k G)^+$ and $(F \circ^k G)^+$ of S are defined as follows:

$$F^{+k} : S \longrightarrow [0,1] | x \longmapsto F^k(x) = F(x) \vee \frac{1-k}{2},$$

$$(F \wedge^k G)^+ : S \longrightarrow [0,1] | x \longmapsto (F \wedge^k G)(x) = (F \wedge G)(x) \vee \frac{1-k}{2},$$

$$(F \vee^k G)^+ : S \longrightarrow [0,1] | x \longmapsto (F \vee^k G)(x) = (F \vee G)(x) \vee \frac{1-k}{2},$$

$$(F \circ^k G)^+ : S \longrightarrow [0,1] | x \longmapsto (F \circ^k G)(x) = (F \circ G)(x) \vee \frac{1-k}{2}.$$

Lemma 4.2. *Let F and G be fuzzy subsets of S. Then the following hold:*

(i) $\left(F \wedge^k G\right)^+ = \left(F^{+k} \wedge G^{+k}\right)$,

(ii) $\left(F \vee^k G\right)^+ = \left(F^{+k} \vee G^{+k}\right)$,

(iii) $\left(F\circ^k G\right)^+ \succeq \left(F^{+k}\circ G^{+k}\right)$, *if* $A_x=\emptyset$ *and* $\left(F\circ^k G\right)^+ = \left(F^{+k}\circ G^{+k}\right)$, *if* $A_x\neq\emptyset$.

Proof. (i) and (ii) follows from [10, Proposition 13].

(iii) Let $a\in S$. If $A_a=\emptyset$, then $\left(F\circ^k G\right)^+(a)=(F\circ G)(a)\vee\frac{1-k}{2}=0\vee\frac{1-k}{2}=\frac{1-k}{2}$. On the other hand, $\left(F^{+k}\circ G^{+k}\right)(a)=0$ and hence $\left(F^{+k}\circ G^{+k}\right)\preceq\left(F\circ^k G\right)^+$. Let $A_a\neq\emptyset$, then

$$
\begin{aligned}
\left(F\circ^k G\right)^+(a) &= (F\circ G)(a)\vee\frac{1-k}{2}\\
&= \left(\bigvee_{(y,z)\in A_a}(F(y)\wedge G(z))\right)\vee\frac{1-k}{2}\\
&= \bigvee_{(y,z)\in A_a}((F(y)\wedge G(z)))\vee\frac{1-k}{2}\\
&= \bigvee_{(y,z)\in A_a}\left(\left(F(y)\vee\frac{1-k}{2}\right)\wedge\left(G(z)\vee\frac{1-k}{2}\right)\right)\\
&= \bigvee_{(y,z)\in A_a}\left(F^{+k}(y)\wedge G^{+k}(z)\right)\\
&= \left(F^{+k}\circ G^{+k}\right)(a).
\end{aligned}
$$
□

Let A be a non-empty subset of S, then the upper part of the characteristic function χ_A^k is defined as follows:

$$\chi_A^{+k}:S\longrightarrow[0,1]|x\longmapsto\chi_A^{+k}(x)=\begin{cases}1, & \text{if } x\in A,\\ \frac{1-k}{2}, & \text{otherwise.}\end{cases}$$

Lemma 4.3. *Let A and B be non-empty subset of S. Then the following hold:*

(i) $(\chi_A\wedge^k\chi_B)^+=\chi_{A\cap B}^{+k}$,

(ii) $(\chi_A\vee^k\chi_B)^+=\chi_{A\cup B}^{+k}$,

(iii) $(\chi_A\circ^k\chi_B)^+=\chi_{(AB]}^{+k}$.

Proof. The proofs of (i) and (ii) are obvious.

(iii) Let $x\in(AB]$ then $\chi_{(AB]}(x)=1$ and hence $\chi_{(AB]}^{+k}(x)=1\vee\frac{1-k}{2}=1$. Since $x\in(AB]$, we have $x\leq ab$ for some $a\in A$ and $b\in B$. Then

$(a,b) \in A_x$ and $A_x \neq \emptyset$. Thus

$$\begin{aligned}\left(\chi_A \circ^k \chi_B\right)^+(x) &= (\chi_A \circ \chi_B)(x) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z)\in A_x} (\chi_A(y) \wedge \chi_B(z))\right] \vee \frac{1-k}{2} \\ &\geq (\chi_A(a) \wedge \chi_B(b)) \vee \frac{1-k}{2}.\end{aligned}$$

Since $a \in A$ and $b \in B$, we have $\chi_A(a) = 1$ and $\chi_B(b) = 1$ and so

$$\begin{aligned}(\chi_A \circ^k \chi_B)^+(x) &\geq (\chi_A(a) \wedge \chi_B(b)) \vee \frac{1-k}{2} \\ &= (1 \wedge 1) \vee \frac{1-k}{2} = 1.\end{aligned}$$

Thus $(\chi_A \circ^k \chi_B)^+(x) = 1 = \chi^{+k}_{(AB]}(x)$. Let $x \notin (AB]$, then $\chi_{(AB]}(x) = 0$ and hence, $\chi^{+k}_{(AB]}(x) = 0 \vee \frac{1-k}{2} = \frac{1-k}{2}$. Let $(y,z) \in A_x$. Then

$$\begin{aligned}(\chi_A \circ^k \chi_B)^+(x) &= (\chi_A \circ \chi_B)(x) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z)\in A_x} (\chi_A(y) \wedge \chi_B(z))\right] \vee \frac{1-k}{2}.\end{aligned}$$

Since $(y,z) \in A_x$, then $x \leq yz$. If $y \in A$ and $z \in B$, we have $yz \in AB$ and so $x \in (AB]$. This is a contradiction. If $y \notin A$ and $z \in B$, then

$$\left[\bigvee_{(y,z)\in A_x} (\chi_A(y) \wedge \chi_B(z))\right] \vee \frac{1-k}{2} = \left[\bigvee_{(y,z)\in A_x} (0 \wedge 1)\right] \vee \frac{1-k}{2} = \frac{1-k}{2}.$$

Hence, $\chi^{+k}_{(AB]}(x) = \frac{1-k}{2} = (\chi_A \circ^k \chi_B)^+(x)$. Similarly, for $y \in A$ and $z \notin B$, we have $\chi^{+k}_{(AB]}(x) = 0 = (\chi_A \circ^k \chi_B)^+(x)$. □

Lemma 4.4. *The upper part χ^{+k}_A of the characteristic function χ_A of A is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S if and only if A is a generalized bi-ideal of S.*

Proof. Let A be a generalized bi-ideal of S. Then by Theorem 2.4 and 3.19, χ^{+k}_A is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S. Conversely, assume that χ^{+k}_A is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S. Let $x, y \in S$, $x \leq y$. If $y \in A$, then $\chi^{+k}_A(y) = 1$. Since χ^{+k}_A is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S and $x \leq y$, we have, $\chi^{+k}_A(x) \geq \chi^{+k}_A(y) = 1$.

It follows that $\chi_A^{+k}(x) = 1$ and so $x \in A$. Let $x, z \in A$ and $y \in S$. Then, $\chi_A^{+k}(x) = 1$ and $\chi_A^{+k}(z) = 1$. Now,

$$\chi_A^{+k}(xyz) \geq \chi_A^{+k}(x) \wedge \chi_A^{+k}(z) = 1.$$

Hence $\chi_A^{+k}(xyz) = 1$ and so $xyz \in A$. Therefore A is a generalized bi-ideal of S. □

Lemma 4.5. *The upper part χ_A^{+k} of the characteristic function χ_A of A is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left (right)-ideal of S if and only if A is a left (right)-ideal of S.*

Proof. The proof follows from Lemma 4.4. □

In the following proposition, we show that if F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S, then F^{+k} is a fuzzy generalized bi-ideal of S.

Proposition 4.6. *If F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S, then F^{+k} is a fuzzy generalized bi-ideal of S.*

Proof. Let $x, y \in S$, $x \leq y$. Since F is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S and $x \leq y$, then we have $F^{+k}(x) = F(x) \vee \frac{1-k}{2} \geq F(y)$. It follows that $F^{+k}(x) = F(x) \vee \frac{1-k}{2} = \left(F(x) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2} \geq F(y) \vee \frac{1-k}{2} = F^{+k}(y)$.

For $x, y, z \in S$, we have $F^{+k}(xyz) = F(xyz) \vee \frac{1-k}{2} \geq F(x) \wedge F(z)$. Then

$$\begin{aligned} F^{+k}(xyz) &= F(xyz) \vee \frac{1-k}{2} = \left(F(xyz) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2} \\ &\geq (F(x) \wedge F(z)) \vee \frac{1-k}{2} \\ &= \left(F(x) \vee \frac{1-k}{2}\right) \wedge \left(F(z) \vee \frac{1-k}{2}\right) \\ &= F^{+k}(x) \wedge F^{+k}(z). \end{aligned}$$

Consequently, F^{+k} is a fuzzy generalized bi-ideal of S. □

In [33], regular and left weakly regular ordered semigroups are characterized by the properties of their fuzzy left (right) and fuzzy generalized bi-ideals. In the following we characterize regular, left weakly regular, left and right simple and completely regular ordered semigroups in terms of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left (right) and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideals. We first need the following result.

Proposition 4.7. ([23]) *An ordered semigroup S is left (right) simple if and only if $(Sa] = S$ $((aS] = S)$ for every $a \in S$.*

Proposition 4.8. *If S is regular, left and right simple ordered semigroup then for every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal F of S we have $F^{+k}(a) = F^{+k}(b)$, for every $a, b \in S$.*

Proof. Assume that S is regular, left and right simple and F a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S. We consider, $E_S = \{e \in S | e \leq e^2\}$, then E_S is obviously non-empty. Let $s, t \in E_S$. Since S is left and right simple, by Proposition 4.7, it follows that $S = (Ss]$ and $S = (tS]$. Since $t \in S$, we have $t \in (Ss]$ and $t \in (sS]$, then $t \leq xs$ and $t \leq sy$ for some $x, y \in S$, and we have

$$t^2 \leq (sy)(xs) = s(yx)s.$$

Since F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal of S, we have

$$\begin{aligned} F(t^2) \vee \frac{1-k}{2} &\geq F(s(yx)s) \vee \frac{1-k}{2} \\ &\geq (F(s) \wedge F(s)) \\ &= F(s). \end{aligned}$$

Thus $F^{+k}(t^2) = F(t^2) \vee \frac{1-k}{2} = \left(F(t^2) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2} \geq F(s) \vee \frac{1-k}{2} = F^{+k}(s)$ and we have

$$\text{(I)} \qquad \overset{+k}{F}(t^2) \geq \overset{+k}{F}(s).$$

Since $t \in E_S$, we have $t \leq t^2$ and so $F(t) \vee \frac{1-k}{2} \geq F(t^2)$. It follows that

$$\begin{aligned} F^{+k}(t) &= \left(F(t) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2} \\ &\geq F(t^2) \vee \frac{1-k}{2}, \end{aligned}$$

and so $F^{+k}(t) \geq F^{+k}(t^2)$. Thus, by (I), we have $F^{+k}(t) \geq F^{+k}(s)$. On the other hand, since $t \in S$, by Proposition 4.7, we have $(St] = S = (tS]$. Since $s \in S$, we have $s \in (St]$ and $s \in (tS]$, then $s \leq at$ and $s \leq tb$ for some $a, b \in S$. Thus, by the same arguments as above, we get $F^{+k}(s) \geq F^{+k}(t)$. It follows that $F^{+k}(t) = F^{+k}(s)$ and hence F^{+k} is constant on E_S.

Now, let $a \in S$. Then there exists $x \in S$ such that $a \leq axa$. It follows that

$$\begin{aligned} ax &\leq (axa)x \\ &= (ax)(ax) \\ &= (ax)^2, \end{aligned}$$

and

$$\begin{aligned} xa &\leq x(axa) \\ &= xa)(xa) \\ &= (xa)^2. \end{aligned}$$

Thus, $ax, xa \in E_S$. By previous arguments, we get, $F^{+k}(ax) = F^{+k}(b) = F^{+k}(xa)$. Since $(ax)a(xa) = (axa)xa \geq axa \geq a$, we have

$$\begin{aligned} F^{+k}(a) &= F(a) \vee \frac{1-k}{2} \geq F((ax)a(xa)) \vee \frac{1-k}{2} \\ &\geq (F(ax) \wedge F(xa)). \end{aligned}$$

Thus

$$\begin{aligned} F^{+k}(a) &= F(a) \vee \frac{1-k}{2} = \left(F(a) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2} \\ &\geq (F(ax) \wedge F(xa)) \vee \frac{1-k}{2} \\ &= \left(F(ax) \vee \frac{1-k}{2}\right) \wedge \left(F(xa) \vee \frac{1-k}{2}\right), \end{aligned}$$

and thus $F^{+k}(a) \geq F^{+k}(ax) \wedge F^{+k}(xa) = F^{+k}(b)$. Since $b \in (Sa]$ and $b \in (aS]$, we have $b \leq pa$ and $b \leq aq$ for some $p, q \in S$. Then $b^2 \leq (aq)(pa) = a(qp)a$ and thus

$$\begin{aligned} F(b^2) \vee \frac{1-k}{2} &\geq F(a(qp)a) \\ &\geq (F(a) \wedge F(a)) \\ &= F(a). \end{aligned}$$

Hence, $F^{+k}(b^2) = F(b^2) \vee \frac{1-k}{2} = \left(F(b^2) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2} \geq F(a) \vee \frac{1-k}{2}$ and we have, $F^{+k}(b^2) \geq F^{+k}(a)$. Since $b \in E_S$, $b^2 \geq b$, then $F(b) \vee \frac{1-k}{2} \geq F(b^2)$ and hence $\overset{+k}{F}(b) = F(b) \vee \frac{1-k}{2} = \left(F(b) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2} \geq F(b^2) \vee \frac{1-k}{2}$, it follows that $F^{+k}(b) \geq F^{+k}(b^2)$ and so $F^{+k}(b) \geq F^{+k}(a)$. Thus, $F^{+k}(b) = F^{+k}(a)$ and so, F^{+k} is a constant function on S. □

Proposition 4.9. ([23]) *An ordered semigroup S is completely regular if and only if for every $A \subseteq S$, we have $A \subseteq (A^2SA^2]$.*

Proposition 4.10. *Let S be a completely regular ordered semigroup. Then for every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal F of S, we have*

$$F^{+k}(a) = F^{+k}(a^2) \text{ for every } a \in S.$$

Proof. Let $a \in S$. Since S is completely regular, by Proposition 4.9, $a \in (a^2Sa^2]$. Then there exists $x \in S$ such that $a \leq a^2xa^2$. Since F is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal of S, we have

$$\begin{aligned} F(a) \vee \frac{1-k}{2} &\geq F(a^2xa^2) \vee \frac{1-k}{2} \\ &\geq \big(F(a^2) \wedge F(a^2)\big) \\ &= F(a^2) \\ &\geq (F(a) \wedge F(a)) \\ &= F(a). \end{aligned}$$

Thus, $F^{+k}(a) = F(a) \vee \frac{1-k}{2} = \big(F(a) \vee \frac{1-k}{2}\big) \vee \frac{1-k}{2} \geq F(a^2) \vee \frac{1-k}{2} \geq F(a) \vee \frac{1-k}{2}$, and it follows that $F^{+k}(a) \geq F^{+k}(a^2) \geq F^{+k}(a)$. Thus $F^{+k}(a) = F^{+k}(a^2)$ for every $a \in S$. □

Theorem 4.11. *If every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal F of S satisfies the condition, $F^{+k}(t) = F^{+k}(t^2)$ for every $t \in S$. Then S is completely regular.*

Proof. Let $t \in S$. We consider the generalized bi-ideal $B(t^2) = (t^2 \cup t^2St^2]$ of S, generated by $t^2 (t \in S)$. Then by Lemma 4.4,

$$\overset{+k}{\chi}_{B(t^2)}(t) = \begin{cases} 1, & \text{if } t \in B(t^2), \\ \frac{1-k}{2}, & \text{otherwise}, \end{cases}$$

is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal of S. By hypothesis, we have

$$\overline{\chi}^k_{B(t^2)}(t^2) = \overline{\chi}^k_{B(t^2)}(t).$$

Since $t^2 \in B(t^2)$, we have $\chi^{+k}_{B(t^2)}(t^2) = 1$ and hence, $\chi^{+k}_{B(t^2)}(t) = 1$, thus $t \in B(t^2)$ and hence, $t \leq t^2$ or $t \leq t^2xt^2$. If $t \leq t^2$, then $t \leq t^2 = tt \leq t^2t^2 = ttt^2 \leq t^2tt^2 \in t^2St^2$ and $t \in (t^2St^2]$. If $t \leq t^2xt^2$, then $t \in t^2St^2$ and $t \in (t^2St^2]$. Thus, S is completely regular. □

Proposition 4.12. ([22]) *Let S be an ordered semigroup, then the following conditions are equivalent:*

(i) S is regular,

(ii) $B \cap L \subseteq (BL]$ for every generalized bi-ideal B and left ideal L of S,

(iii) $B(a) \cap L(a) \subseteq (B(a)L(a)]$ for every $a \in S$.

Theorem 4.13. *An ordered semigroup S is regular if and only if for every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal F and $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy left* *ideal G of S, we have*

$$\left(F \wedge^k G\right)^+ \preceq \left(F \circ^k G\right)^+ .$$

Proof. Suppose that F is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal and G a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy left ideal of a regular ordered semigroup S. Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq axa \leq (axa)(xa)$. Then $(axa, xa) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}
\left(F \circ^k G\right)^+ (a) &= (F \circ G)(a) \vee \frac{1-k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} (F(y) \wedge G(z))\right] \vee \frac{1-k}{2} \\
&\geq (F(axa) \wedge G(xa)) \vee \frac{1-k}{2} \\
&= \left(F(axa) \vee \frac{1-k}{2}\right) \wedge \left(G(xa) \vee \frac{1-k}{2}\right) \\
&= F^{+k}(axa) \wedge G^{+k}(xa).
\end{aligned}$$

Since F is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal and G an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy left ideal of S, we have $F^{+k}(axa) \geq F^{+k}(a) \wedge F^{+k}(a) = F^{+k}(a)$ and $G^{+k}(xa) \geq G^{+k}(a)$. Therefore,

$$\left[F^{+k}(axa) \wedge G^{+k}(xa)\right] \geq F^{+k}(a) \wedge G^{+k}(a) = \left(F \wedge^k G\right)^+ (a).$$

Thus $\left(F \circ^k G\right)^+ (a) \geq \left(F \wedge^k G\right)^+ (a)$.

Conversely, assume that $\left(F \wedge^k G\right)^+ \preceq \left(F \circ^k G\right)^+$ for every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal F and every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy left ideal G of S. To prove that S is regular, by Proposition 4.12 it is enough to prove that,

$$B \cap L \subseteq (BL] \text{ for generalized bi-ideal } B \text{ and left ideal } L \text{ of } S.$$

Let $x \in B \cap L$. Then $x \in B$ and $x \in L$. Since B is a generalized bi-ideal and L a left ideal of S, by Lemma 4.4 and 4.5, χ_B^{+k} is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal and χ_L^{+k} a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy left ideal of S. By hypothesis, $\left(\chi_B \circ^k \chi_L\right)^+ (x) \geq (\chi_B \wedge^k \chi_L)^+(x) = (\chi_B^k \wedge \chi_L^k)(x) \vee \frac{1-k}{2}$. Since $x \in B$ and $x \in L$, we have $\chi_B^{+k}(x) = 1$ and $\chi_L^{+k}(x) = 1$. Thus $\left(\chi_B^{+k} \wedge \chi_L^{+k}\right)(x) = 1$. It follows that $\left(\chi_B \circ^k \chi_L\right)^+ (x) = 1$. By using Lemma 4.3 (iii), we have $\left(\chi_B \circ^k \chi_L\right)^+ = \chi_{(BL]}^{+k}$. Therefore, $\chi_{(BL]}^{+k}(x) = 1$ and so $x \in (BL]$. Consequently, S is regular. □

Proposition 4.14. ([22]) *Let S be an ordered semigroup, then the following conditions are equivalent:*
(i) S is regular,
(ii) $B \cap I = (BIB]$ for every generalized bi-ideal B and ideal I of S,
(iii) $B(a) \cap I(a) = (B(a) I(a) B(a)]$ for every $a \in S$.

Theorem 4.15. *An ordered semigroup S is regular if and only if for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal F and every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy ideal G of S, we have*

$$\left(F \wedge^k G\right)^+ \preceq \left(F \circ^k G \circ^k F\right)^+.$$

Proof. Suppose that F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy generalized bi-ideal and G a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy ideal of a regular ordered semigroup S. Let $a \in S$. Since S is regular, there exits $x \in S$ such that $a \leq axa \leq (axa)(xa) = a(xaxa)$. Then $(a, xaxa) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}
& \left(F \circ^k G \circ^k F\right)^+(a) \\
= \; & (F \circ^k G \circ F)(a) \vee \frac{1-k}{2} \\
= \; & \left[\bigvee_{(y,z)\in A_a} (F \circ^k G)(y) \wedge F(z)\right] \vee \frac{1-k}{2} \\
= \; & \bigvee_{(y,z)\in A_a} \left[\bigvee_{(p,q)\in A_a} \left((F(p) \wedge G(q)) \vee \frac{1-k}{2}\right) \wedge F(z)\right] \vee \frac{1-k}{2} \\
= \; & \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_a} ((F(p) \wedge G(q)) \wedge F(z)) \vee \frac{1-k}{2} \\
= \; & \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_a} (F(p) \wedge G(q) \wedge F(z)) \vee \frac{1-k}{2} \\
\geq \; & (F(a) \wedge G(xax) \wedge F(a)) \vee \frac{1-k}{2}.
\end{aligned}$$

Since G a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy ideal of S, we have $G(xax) \vee \frac{1-k}{2} \geq G(ax) \wedge \frac{1-k}{2} \geq G(a)$. Therefore

$$\begin{aligned}
& \left[F(a) \wedge G(xax) \wedge F(a) \wedge \frac{1-k}{2}\right] \\
= \; & \left[F(a) \wedge \left(G(xax) \vee \frac{1-k}{2}\right) \wedge F(a) \vee \frac{1-k}{2}\right] \\
\geq \; & \left[F(a) \wedge G(a) \wedge F(a) \vee \frac{1-k}{2}\right]
\end{aligned}$$

$$\begin{aligned} &\geq \left(F(a) \vee \frac{1-k}{2}\right) \wedge \left(G(a) \vee \frac{1-k}{2}\right) \\ &= F^{+}(a) \wedge^{k} G^{+}(a). \end{aligned}$$

Thus $\left(F \circ^k G \circ^k F\right)^{+}(a) \geq \left(F \wedge^k G\right)^{+}(a)$.

Conversely, assume that $\left(F \wedge^k G\right)^{+} \preceq \left(F \circ^k G \circ^k F\right)^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal F and every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy ideal F of S. To prove that S is regular, by Proposition 4.14, it is enough to prove that $B \cap I \subseteq (BIB]$ for generalized bi-ideal B and ideal I of S.

Let $x \in B \cap I$. Then $x \in B$ and $x \in I$. Since B is a generalized bi-ideal and I an ideal of S, by Lemma 4.4 and 4.5, χ_B^{+k} is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal and χ_I^{+k} an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy ideal of S. By hypothesis, $(\chi_B \circ^k \chi_I \circ^k \chi_B)^{+}(x) \geq (\chi_B \wedge^k \chi_I)^{+}(x) = (\chi_B \wedge \chi_I)(x) \vee \frac{1-k}{2}$. Since $x \in B$ and $x \in I$, we have $\chi_B^{+k}(x) = 1$ and $\chi_I^{+k}(x) = 1$. Thus $(\chi_B \wedge \chi_I)(x) \vee \frac{1-k}{2} = 1$. It follows that $(\chi_B \circ^k \chi_I \circ^k \chi_B)^{+}(x) = 1$. By using Lemma 4.3 (iii), we have $(\chi_B \circ^k \chi_I \circ^k \chi_B)^{+}(x) = \chi_{(BIB]}^{+k}(x)$. Therefore, $\chi_{(BIB]}^{+k}(x) = 1$ and so $x \in (BIB]$. Consequently, S is regular. □

Proposition 4.16. ([22]) *Let S be an ordered semigroup, then the following conditions are equivalent:*

(i) S is regular,

(ii) $R \cap B \cap L \subseteq (RBL]$ for every right ideal R, generalized bi-ideal B and left ideal L of S,

(iii) $R(a) \cap B(a) \cap L(a) \subseteq (R(a) B(a) L(a)]$ for every $a \in S$.

Theorem 4.17. *An ordered semigroup S is regular if and only if for every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal F, every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy right ideal G and every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy left ideal H of S, we have,*

$$(F \wedge G \wedge H)^{+} \preceq (G \circ G \circ H)^{+}.$$

Proof. Let S be a regular ordered semigroup, F a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal, G a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy right ideal and H a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy left ideal of S. Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq axa = axa \leq (axa)(xa) \leq (axa)x(axa) = (ax)(axa)(xa)$. Then $((ax)(axa), xa) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned} &\left(F \circ^k G \circ^k H\right)^{+}(a) \\ &= (F \circ^k G \circ H)(a) \vee \frac{1-k}{2} \end{aligned}$$

$$
\begin{aligned}
&= \left[\bigvee\nolimits_{(y,z)\in A_a}(F\circ^k G)(y)\wedge H(z)\right]\vee\frac{1-k}{2}\\
&= \bigvee\nolimits_{(y,z)\in A_a}\left[\bigvee\nolimits_{(p,q)\in A_a}\left((F(p)\wedge G(q))\vee\frac{1-k}{2}\right)\wedge H(z)\right]\vee\frac{1-k}{2}\\
&= \bigvee\nolimits_{(y,z)\in A_a}\bigvee\nolimits_{(p,q)\in A_a}((F(p)\wedge G(q))\wedge H(z))\vee\frac{1-k}{2}\\
&= \bigvee\nolimits_{(y,z)\in A_a}\bigvee\nolimits_{(p,q)\in A_a}(F(p)\wedge G(q)\wedge H(z))\vee\frac{1-k}{2}\\
&\geq (F(ax)\wedge G(axa)\wedge H(xa))\vee\frac{1-k}{2}.
\end{aligned}
$$

Since F is a $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_k)$-fuzzy right ideal G, a $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_k)$-fuzzy generalized ideal and H a $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_k)$-fuzzy left ideal of S, we have $F(ax)\vee\frac{1-k}{2}\geq F(a)\vee\frac{1-k}{2}$, $G(axa)\vee\frac{1-k}{2}\geq G(a)\wedge G(a)\vee\frac{1-k}{2}$ and $H(xa)\vee\frac{1-k}{2}\geq H(a)\vee\frac{1-k}{2}$. Therefore,

$$
\begin{aligned}
&[F(ax)\wedge G(axa)\wedge H(xa)]\vee\frac{1-k}{2}\\
&= \left[\left(F(ax)\vee\frac{1-k}{2}\right)\wedge\left(G(axa)\vee\frac{1-k}{2}\right)\wedge\left(H(xa)\vee\frac{1-k}{2}\right)\right]\\
&\geq \left[\left(F(a)\vee\frac{1-k}{2}\right)\wedge\left(G(a)\vee\frac{1-k}{2}\right)\wedge\left(H(a)\vee\frac{1-k}{2}\right)\right]\\
&= F^+(a)\wedge^k G^+(a)\wedge^k H^+(a).
\end{aligned}
$$

Thus $\left(F\circ^k G\circ^k H\right)^+(a)\geq\left(F\wedge^k G\wedge^k H\right)^+(a)$.

Conversely, assume that $\left(\lambda\wedge^k\mu\wedge^k\rho\right)^+\preceq\left(\lambda\circ^k\mu\circ^k\rho\right)^+$ for every $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_k)$-fuzzy right ideal F, every $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_k)$-fuzzy generalized ideal G and every $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_k)$-fuzzy left ideal H of S. To prove that S is regular, by Proposition 4.16, it is enough to prove that $R\cap B\cap L\subseteq(RBL]$ for right ideal R, generalized bi-ideal B and left ideal L of S. Let $x\in R\cap B\cap L$. Then $x\in R$, $x\in B$ and $x\in L$. Since R is a right ideal, B a generalized bi-ideal and L a left ideal of S, by Lemma 4.4 and 4.5, χ_R^{+k} is an $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_k)$-fuzzy right ideal, χ_B^{+k} an $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal and χ_L^{+k} an $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_k)$-fuzzy left ideal of S. By hypothesis, $(\chi_R\circ^k\chi_B\circ^k\chi_L)^+(x)\geq(\chi_R\wedge^k\chi_B\wedge^k\chi_L)^+(x)=(\chi_R\wedge\chi_B\wedge\chi_L)(x)\vee\frac{1-k}{2}$. Since $x\in R$, $x\in B$ and $x\in L$, we have $\chi_R^{+k}=1$, $\chi_B^{+k}=1$ and $\chi_L^{+k}=1$. Thus $(\chi_R\wedge^k\chi_B\wedge^k\chi_L)^+(x)=(\chi_R\wedge\chi_B\wedge\chi_L)(x)\vee\frac{1-k}{2}=1$. Follows that $(\chi_R\circ^k\chi_B\circ^k\chi_L)^+(x)=1$. By using Lemma 4.4 (iii), we have $(\chi_R \circ^k \chi_B \circ^k \chi_L)^+ = \chi^{+k}_{(RBL]}$. Therefore, $\chi^{+k}_{(RBL]}(x) = 1$ and so $x \in (RBL]$. Consequently, S is regular. □

Proposition 4.18. ([22]) *Let $(S, \cdot, \leq)$ be an ordered semigroup, then the following conditions are equivalent:*
(1) S is left weakly regular,
(2) $I \cap L \subseteq (IL]$ for every ideal I and left ideal L of S,
(3) $I(a) \cap L(a) \subseteq (I(a)L(a)]$ for every $a \in S$.

Theorem 4.19. *An ordered semigroup $(S, \cdot, \leq)$ is left weakly regular if and only if for every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy ideal F and every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left ideal G of S,*

$$(F \wedge^k G)^+ \preceq (F \circ^k G)^+.$$

Proof. Suppose that F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy ideal and G a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left ideal of a left weakly regular ordered semigroup S. Let $a \in S$. Since S is left weakly regular, there exists $x, y \in S$ such that $a \leq xaya = (xa)(ya)$. Then $(xa, ya) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned} \left(F \circ^k G\right)^+(a) &= (F \circ G)(a) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z)\in A_a} (F(y) \wedge G(z))\right] \vee \frac{1-k}{2} \\ &\geq (F(xa) \wedge G(ya)) \vee \frac{1-k}{2}. \end{aligned}$$

Since F is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy ideal and G an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left ideal of S, we have $F(xa) \vee \frac{1-k}{2} \geq F(a) \vee \frac{1-k}{2}$ and $G(ya) \vee \frac{1-k}{2} \geq G(a) \vee \frac{1-k}{2}$. Therefore,

$$\begin{aligned} (F(xa) \wedge G(ya)) \vee \frac{1-k}{2} &= \left[\left(F(xa) \vee \frac{1-k}{2}\right) \wedge \left(G(ya) \vee \frac{1-k}{2}\right)\right] \\ &\geq \left(F(a) \vee \frac{1-k}{2}\right) \wedge \left(G(a) \vee \frac{1-k}{2}\right) \\ &= F^+(a) \wedge^k G^+(a). \end{aligned}$$

Thus $\left(F \circ^k G\right)^+(a) \geq \left(F \wedge^k G\right)^+(a)$.

Conversely, assume that $\left(\lambda \wedge^k \mu\right)^- \preceq \left(\lambda \circ^k \mu\right)^-$ for every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy ideal λ and every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy left ideal μ of S. To prove that S is left weakly regular, by Proposition 4.18, it is enough to prove that

$$I \cap L \subseteq (IL] \text{ for ideal } I \text{ and left ideal } L \text{ of } S.$$

Let $x \in I \cap L$. Then $x \in I$ and $x \in L$. Since I is an ideal and L a left ideal of S, by Lemma 4.4 and 4.5, χ_I^{+k} is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy ideal and $\overline{\chi}_L^k$ an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy left ideal of S. By hypothesis, $(\chi_I \circ^k \chi_L)^+(x) \geq (\chi_I \wedge^k \chi_L)^+(x) = (\chi_I^k \wedge \chi_L^k)(x) \vee \frac{1-k}{2}$. Since $x \in I$ and $x \in L$, we have $\chi_I^{+k} = 1$ and $\chi_L^{+k} = 1$. Thus $(\chi_I^{+k} \wedge \chi_L^{+k})(x) \vee \frac{1-k}{2} = 1$. It follows that $(\chi_I \circ^k \chi_L)^+(x) = 1$. By using Lemma 4.3 (iii), we have $(\chi_I \circ^k \chi_L)^+ = \chi_{(IL]}^{+k}(x)$. Therefore, $\chi_{(IL]}^{+k}(x) = 1$ and so $x \in (IL]$. Consequently, S is left weakly regular. □

Proposition 4.20. ([22]) *Let S be an ordered semigroup, then the following conditions are equivalent:*

(i) S is left weakly regular,

(ii) $I \cap B \subseteq (IB]$ for every generalized bi-ideal B and every ideal I of S,

(iii) $I(a) \cap B(a) \subseteq (I(a)B(a)]$ for every $a \in S$.

Theorem 4.21. *An ordered semigroup $(S, \cdot, \leq)$ is left weakly regular if and only if for every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy ideal F and every $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal G of S,*

$$(F \wedge^k G)^+ \preceq (F \circ^k G)^+.$$

Proof. Suppose that F is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy ideal and G a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal of a left weakly regular ordered semigroup S. Let $a \in S$. Since S is left weakly regular, there exists $x, y \in S$ such that $a \leq xaya \leq x(xaya)ya = (x^2ay)(aya)$. Then $\left(x^2ay, aya\right) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}\left(F \circ^k G\right)^+(a) &= (F \circ G)(a) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z)\in A_a} (F(y) \wedge G(z))\right] \vee \frac{1-k}{2} \\ &\geq \left(F\left(x^2ay\right) \wedge G(aya)\right) \vee \frac{1-k}{2}.\end{aligned}$$

Since F is a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy ideal and G a $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy generalized bi-ideal of S, we have $F\left(x^2ay\right) \vee \frac{1-k}{2} \geq F(ay) \vee \frac{1-k}{2} \geq F(a) \vee \frac{1-k}{2}$ and $G(aya) \vee \frac{1-k}{2} \geq (G(a) \wedge G(a)) \vee \frac{1-k}{2} = G(a) \vee \frac{1-k}{2}$. Therefore

$$\begin{aligned}&F\left(x^2ay\right) \wedge G(aya) \vee \frac{1-k}{2} \\ =\ &\left[\left(F\left(x^2ay\right) \vee \frac{1-k}{2}\right) \wedge \left(G(aya) \vee \frac{1-k}{2}\right)\right]\end{aligned}$$

$$\begin{aligned} &\geq \left(F\left(a\right)\wedge\frac{1-k}{2}\right)\wedge\left(G\left(a\right)\wedge\frac{1-k}{2}\right)\\ &= F^{+}\left(a\right)\wedge^{k}G^{+}\left(a\right)\end{aligned}$$

Thus $\left(F\circ^{k}G\right)^{+}(a)\geq\left(F\wedge^{k}G\right)^{+}(a)$.

Conversely, assume that $\left(\lambda\wedge^{k}\mu\right)^{+}\preceq\left(\lambda\circ^{k}\mu\right)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_{k})$-fuzzy ideal F and every $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_{k})$-fuzzy generalized bi-ideal G of S. To prove that S is left weakly regular, by Proposition 4.20, it is enough to prove that

$$I\cap B\subseteq(IL]\text{ for ideal }I\text{ and generalized bi-ideal }B\text{ of }S.$$

Let $x\in I\cap B$. Then $x\in I$ and $x\in B$. Since I is an ideal and B a generalized bi-ideal of S, by Lemma 4.4 and 4.5, χ_{I}^{+k} is an $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_{k})$-fuzzy ideal and χ_{B}^{+k} an $(\overline{\in},\overline{\in}\vee\overline{\mathrm{q}}_{k})$-fuzzy generalized bi-ideal of S. By hypothesis, $(\chi_{I}\circ^{k}\chi_{B})^{+}(x)\geq(\chi_{I}\wedge^{k}\chi_{B})^{+}(x)=(\chi_{I}\wedge\chi_{B})(x)\vee\frac{1-k}{2}$. Since $x\in I$ and $x\in B$, we have $\chi_{I}^{+k}(x)=1$ and $\chi_{B}^{+k}(x)=1$. Thus $(\chi_{I}\wedge\chi_{B})(x)\vee\frac{1-k}{2}=1$. It follows that $(\chi_{I}\circ^{k}\chi_{B})^{+}(x)=1$. By using Lemma 4.3 (iii), we have $(\chi_{I}\circ^{k}\chi_{B})^{+}=\chi_{(IB]}^{+k}$. Therefore, $\chi_{(IB]}^{+k}(x)=1$ and so $x\in(IB]$. Consequently, S is left weakly regular. □

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